Application of Z-Transform for calculating locations and amplitudes of extrema: the case of *sinc* and related functions

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Abstract. *Expressions of the form* $\mathrm{sinc}\, x$, $(\sin x/x)^r$ or $(\sin^2 x)/x$ *may occur in engineering problems, for instance in signal processing or communications. Using Z-Transform techniques, it can be shown that the locations and amplitudes of the extrema of the sinc function, when expressed under the form of series expansions, can be calculated very fastly through a straightforward recursion formula. Moreover, very simple accurate algebraic expressions can be derived for evaluating these locations and amplitudes. Using a similar approach, recursion formulae and algebraic expressions can be obtained for the locations and amplitudes of the extrema of* $(\sin x/x)^r$, *and of those of* $(\sin^2 x)/x$ *as well.*

I. INTRODUCTION

The $\sin x/x$ function (Fig. 1a) plays a key role in the fields of signal processing and communications [1, 2], so its properties have been extensively studied. Its Fourier transform, the rectangular function, is also widely used. On the other hand, the convolution operator and its counterpart regarding the Fourier Transform, the product operator, are of great importance. This paper presents the computation of the extrema of the function $(\sin x/x)^r$, where r is a positive integer. This function can be viewed as the result of the Fourier transform of a product of r identical rectangular pulses. A useful case is that of the function $(\sin x/x)^2$ (Fig. 1b) which represents the Fourier transform of a triangular function.

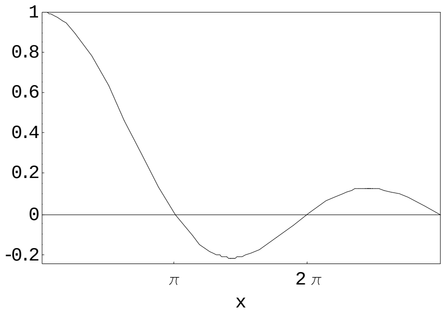

Fig. 1a. $\sin x/x$

For example, a window is frequently used in the time domain to limit the duration of a signal before calculating its Discrete Fourier Tranform. Many kinds of windows can be considered. Generally, the window is chosen with a smooth return to zero at its ends, in order to avoid discontinuity between two successive replicas of the periodized signal. This leads to enhanced properties of its Discrete Fourier Transform. The well known rectangular window does not satisfy this criterion, but its multiple convolutions do. The smoothness at the ends of the so-called polynomial windows [3] increases with the number of convolutions. So the frequential properties of a polynomial window, which corresponds to the convolution of r rectangular windows of duration T, are given by its Fourier Transform proportional to $(\sin x/x)^r$, with $x = \pi T f$. In particular, the triangular window and the parabolic window are relative to $r = 2$ and $r = 3$, respectively.

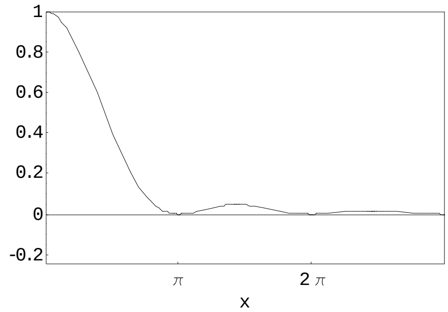

Fig. 1b. $(\sin x/x)^2$

The $(\sin x/x)^r$ function may appear in other applications, for instance when studying random processes [4]. In the case of a simple random binary wave, with pulses of amplitude $+A$ and $-A$ and duration T, the autocorrelation function has a triangular waveform and the expression of the power spectral density , is $A^2T(\sin x/x)^r$, with $x = \pi T f$.

In this paper, it will also be presented the computation of the extrema of the function $y = (\sin^2 x)/x$, which can also be written $y = \sin x$ $(\sin x/x)$ (Fig. 2). Such an expression may be encountered in signal processing or communications problems, and also in other fields of physics or engineering.

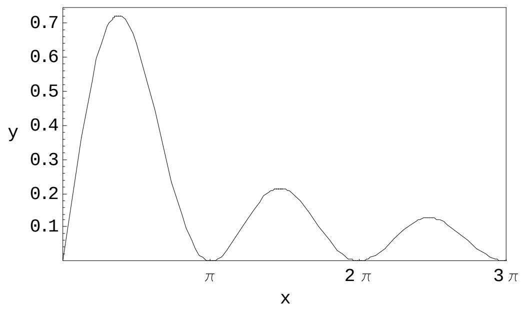

Fig. 2. $y = (\sin^2 x)/x$

For example, if we consider the signal $A(\Pi((t-\theta/2)/\theta)+\Pi((t+\theta/2)/\theta))$ (with $\Pi(t) = 1$ if $|t| \leq 1/2$ and 0 if $|t| > 1/2$), i.e. the sum of two rectangular pulses symmetrical towards the origin (or the convolution of a rectangular pulse $A\Pi(t/\theta)$ by two Dirac impulses $\delta(t-\theta/2)$ and $\delta(t+\theta/2)$ symmetrical towards the origin), its Fourier transform is expressed as follows : $2jA\theta\sin^2(\pi f\theta)/(\pi f\theta)$.

II. EXTREMA OF $(sinx/x)^r$

II.1. Equation

It is clear that the zeros of the function $(\sin x/x)^r$ are at $\pm k\pi$ ($k = 1, 2, ...$) and the global maximum, which equals 1, is at $x = 0$. The minima of $(\sin x/x)^r$ for an even r are at $\pm k\pi$ ($k = 1, 2, ...$) and equal 0. But the local extrema $(\sin x/x)^r$ for any r are not at $\tilde{x}_m = \pm(2m+1)\pi/2$ ($m = 1, 2, ...$). Their locations x_m are, for all r, the solutions of

$$x = \tan x \tag{1}$$

Once the x_m obtained, the extrema amplitudes $y_{m,r}$ can be deduced from

$$y_{m,r} = \cos^r x_m \tag{2}$$

or $y_{m,r} = (-1)^{mr}(1 + x_m^2)^{-r/2}$.

In some applications, accurate determination of the extrema is needed [5]. In [5], a closed-form approximation and a recursion formula were proposed to determine the locations of the local extrema, with a good accuracy. In a more recent paper [6], methods were presented and used for the computation of both the locations and the amplitudes of the $sinx/x$ function's extrema. Especially, series expansions were introduced to express the extrema locations and amplitudes, as well as simple algebraic approximations, allowing to reach an excellent accuracy. These results were extended to the function $(\sin x/x)^r$ in [7]. In particular, it was shown how to compute the coefficients of the $y_{m,r}$ series expansions and give approximate algebraic expressions for $y_{m,r}$.

II.2. Locations of the extrema

II.2.1. Series expansion

Let us denote $e_m = \tilde{x}_m - x_m$ the difference between each exact extremum location x_m and its nearest odd multiple of $\pi/2$, i.e. $\tilde{x}_m$. Then let us write e_m under the form of a series expansion in successive powers of $(\tilde{x}_m)^{-1}$: $e_m = \sum_{n=0}^{+\infty} e_m^n (\tilde{x}_m)^{-n}$. Therefrom the expansion coefficients of the extrema locations $x_m = \sum_{n=-1}^{+\infty} x_m^n (\tilde{x}_m)^{-n}$ can be immediately deduced : $x_m^{-1} = \tilde{x}_m$, $x_m^n = -e_m^n$ ($n \geq 0$).

We show now that using basic z-Transform techniques allows to derive the coefficients of the expansions. Changing $\tilde{x}_m$ into z_m for this purpose and leaving out the index m for the coefficients are independent of m, eqn 1 takes the form

$$\tan e = \frac{1}{z - e} \tag{3}$$

The first derivative can be expressed as

$$\frac{de}{dz} = -\frac{1}{(z - e)^2} \tag{4}$$

A second derivation leads to

$$(z - e)\frac{d^2e}{dz^2} - 2(\frac{de}{dz})^2 + 2\frac{de}{dz} = 0 \tag{5}$$

If $F(z)$ and $G(z)$ denote the z-transforms of f_n anf g_n respectively ($f_n \rightharpoonup F(z)$ and $G(z) \rightharpoonup g_n$), note the following inverse z-transforms : $z^{-1}F(z) \leftharpoondown f_{n-1}$, $-z\frac{dF(z)}{dz} \leftharpoondown nf_n$, $F(z)G(z) \leftharpoondown \sum_{j=0}^{n} f_j g_{n-j} = \sum_{j=0}^{n} f_{n-j} g_j$.

Interpreting eqn 5 by using basic z-Transform properties allows the straightforward computation of the expansion coefficients of e through the following relation (where n in e^n denotes an upper index)

$$\begin{gathered} e^n = \frac{1}{n(n-1)}.. \\ .. \sum_{j=1,\ j\ odd}^{n-2} j(2n - j - 1)e^j e^{n-j-1}, \\ n \geq 3,\ n\ odd \\ e^n = 0,\ n \geq 0,\ n\ even \end{gathered} \tag{6}$$

with $e^1 = 1$. Therefore the expansion of x_m begins as follows

$$\begin{aligned} x_m = \tilde{x}_m - (\tilde{x}_m)^{-1} - \tfrac{2}{3}(\tilde{x}_m)^{-3}.. \\ .. - \tfrac{13}{15}(\tilde{x}_m)^{-5} - \tfrac{146}{105}(\tilde{x}_m)^{-7} - ... \end{aligned} \tag{7}$$

II.2.2. Algebraic approximation

By limiting this expansion, simple closed-form approximations are obtained for x_m. Table 1 shows the values obtained for x_1, x_2, x_3 using the successive terms of the series expansion (the table starts with $x_m \simeq \tilde{x}_m$ at rank

$p = 0$ $(n = 2p - 1)$). It can be observed for example that after only two ranks, the maximum relative error of the very simple algebraic approximation $x_m \simeq \tilde{x}_m - (\tilde{x}_m)^{-1} - \frac{2}{3}(\tilde{x}_m)^{-3}$ is 8.96x10^{-5} and it occurs for $m = 1$. The relative error is less than 3.86x10^{-6} for the other values of m. This very simple formula gives results comparable to those of eqn 1 in [5].

Table 1. Successive values for x_1, x_2, x_3

p	x_1	x_2	x_3
0	4.712388980	7.853981634	10.99557429
1	4.500182390	7.726657680	10.90462861
2	4.493811716	7.725281614	10.90412712
3	4.493438770	7.725252614	10.90412173
4	4.493411825	7.725251859	10.90412166
∞	4.493409458	7.725251837	10.90412166

II.3. Amplitudes of the extrema

II.3.1. Series expansion

Using the same approach as for the locations, the extrema themselves can be directly calculated. From eqn 1, their amplitudes are $y_{m,r} = (-1)^{mr} \sin^r e_m$. Assuming that $y_{m,r}$ can be expanded under the form $y_{m,r} = (-1)^{mr} \sum_{n=0}^{+\infty} y_{m,r}^n (\tilde{x}_m)^{-n}$, and changing $\tilde{x}_m$ into z_m, eqns 3 and 4 lead to the simple following equation (m is omitted as above)

$$-(z - e)\frac{d\,|y_r|}{dz} = r\,|y_r| \qquad (8)$$

Interpreting eqn 8 by using basic z-Transform properties allows here the straightforward computation of the expansion coefficients of $(-1)^{mr} y_{m,r}$ through the following relation (where n in y_r^n or e^n denotes an upper index)

$$y_{m,r}^n = \frac{1}{n-r} \sum_{j=r}^{n-1} j e^{n-j-1} y^j, \quad n > r,\ (n-r)\ even \qquad (9)$$

$$y_{m,r}^n = 0,\ n \geq 0,\ (n < r,\ (n-r)\ odd)$$

with $y_{m,r}^r = 1$. Therefore the expansion of $y_{m,r}$ begins as follows, for $r = 1$ and $r = 2$ as examples

$$y_{m,1} = (-1)^m (\tilde{x}_m)^{-1} + \tfrac{1}{2}(\tilde{x}_m)^{-3} .. \quad .. + \tfrac{13}{24}(\tilde{x}_m)^{-5} + \tfrac{61}{80}(\tilde{x}_m)^{-7} + ... \qquad (10)$$

$$y_{m,2} = (\tilde{x}_m)^{-2} + (\tilde{x}_m)^{-4} + .. \quad .. \tfrac{4}{3}(\tilde{x}_m)^{-6} + \tfrac{31}{15}(\tilde{x}_m)^{-8} + ... \qquad (11)$$

III.3.2. Algebraic approximation

By limiting this expansion, simple closed-form approximations are obtained for $y_{m,r}$.

Table 2 shows the values obtained for y_1^1, y_2^1, y_3^1 using the successive terms of the series expansion (the table starts with $y_{m,1} \simeq (-1)^m (\tilde{x}_m)^{-2}$ at rank $p = 0$ $(n = 2p + 1)$). It can be observed for example that after only two ranks, the maximum relative error of the simple algebraic approximation $y_{m,1} \simeq (-1)^m \left[(\tilde{x}_m)^{-1} + \frac{1}{2}(\tilde{x}_m)^{-3} + \frac{13}{24}(\tilde{x}_m)^{-5}\right]$ is 7.34x10^{-5} and it occurs for $m = 1$. The relative error is less than 3.31x10^{-6} for the other values of m.

Table 2. Successive values for y_1^1, y_2^1, y_3^1

p	y_1^1	y_2^1	y_3^1
0	−0.2122065908	0.1273239545	−0.09094568177
1	−0.2169845959	0.1283560036	−0.09132179296
2	−0.2172176874	0.1283741288	−0.09132516305
3	−0.2172324632	0.1283745424	−0.09132520229
4	−0.2172335359	0.1283745532	−0.09132520282
∞	−0.2172336282	0.1283745535	−0.09132520282

Table 3 shows the values obtained for y_1^2, y_2^2, y_3^2 using the successive terms of the series expansion (the table starts with $y_{m,2} \simeq (\tilde{x}_m)^{-2}$ at rank $p = 0$ $(n = 2p+2)$). It can be observed for example that after only two ranks, the maximum relative error of the simple algebraic approximation $y_{m,2} \simeq (\tilde{x}_m)^{-2} + (\tilde{x}_m)^{-4} + \frac{4}{3}(\tilde{x}_m)^{-6}$ is 1.95x10^{-4} and it occurs for $m = 1$. The relative error is less than 8.91x10^{-6} for the other values of m.

Table 3. Successive values for y_1^2, y_2^2, y_3^2

p	y_1^2	y_2^2	y_3^2
0	0.04503163717	0.01621138938	0.008271117032
1	0.04705948552	0.01647419853	0.008339528409
2	0.04718124196	0.01647987920	0.008340282860
3	0.04718974044	0.01648002194	0.008340292533
4	0.04719039121	0.01648002587	0.008340292669
∞	0.04719044923	0.01648002599	0.008340292671

III. EXTREMA OF $(\sin^2 x)/x$

III.1. Equation

It is clear that the zeros of the function y are at $\pm k\pi$ $(k = 0, 1, 2, ...)$. But the extrema are not at $\tilde{x}_m = \pm(2m + 1)\pi/2$ $(m = 1, 2, ...)$. Their locations x_m are the solutions of

$$\tan x = 2x \qquad (12)$$

Once the x_m obtained, the extrema amplitudes y_m can be deduced from

$$y_m = \frac{4x_m}{1+4x_m^2} \tag{13}$$

In some applications, accurate determination of the extrema locations and amplitudes may be needed, as for the $\sin x/x$ function [5]. In [6], methods were presented for the computation of both the locations and the amplitudes of the $sinx/x$ function's extrema. Especially, series expansions were introduced to express the extrema locations and amplitudes, as well as simple algebraic approximations, allowing to reach an excellent accuracy. In this paper, we exploit similar ideas to obtain analog results for the function $(\sin^2 x)/x$. We show how to compute the coefficients of the x_m and y_m series expansions and give approximate algebraic expressions for x_m and y_m.

As the function y is symmetric towards the origin, we will consider $x \geq 0$ in the following.

III.2. Locations of the extrema

III.2.1. Series expansion

Let us denote $e_m = \tilde{x}_m - x_m$ the difference between each exact extremum location x_m and its nearest odd multiple of $\pi/2$, i.e. $\tilde{x}_m$. Then let us write e_m under the form of a series expansion in successive powers of $(\tilde{x}_m)^{-1}$: $e_m = \sum_{n=0}^{+\infty} e_m^n (\tilde{x}_m)^{-n}$. Therefrom the expansion coefficients of the extrema locations $x_m = \sum_{n=-1}^{+\infty} x_m^n (\tilde{x}_m)^{-n}$ can be immediately deduced : $x_m^{-1} = \tilde{x}_m$, $x_m^n = -e_m^n$ $(n \geq 0)$.

We show now that using basic z-Transform techniques allows to derive the coefficients of the expansions. Changing $\tilde{x}_m$ into z_m for this purpose and leaving out the index m for the coefficients are independent of m, eqn 12 takes the form

$$\tan e = \frac{1}{2(z-e)} \tag{14}$$

Calculating the first derivative of e leads to

$$\left[1-4(z-e)^2\right]\frac{de}{dz} = 2 \tag{15}$$

Interpreting eqn 15 by using basic z-Transform properties allows the straightforward computation of the expansion coefficients of e through the following relation (where n in e^n denotes an upper index)

$$e^n = \frac{n-2}{4n}e^{n-2} + \frac{2}{n}\sum_{j=0}^{n-1} j e^j e^{n-1-j}.. \\ .. - \frac{1}{n}\sum_{j=0}^{n-2}\left[\sum_{k=0}^{j} e^k e^{j-k}\right](n-2-j)e^{n-2-j} \tag{16}$$

if n odd $(n \geq 3)$ with $e^1 = 1/2$, and $e^n = 0$, if n even. Therefore the expansion of x_m begins as follows

$$x_m = \tilde{x}_m - \tfrac{1}{2}(\tilde{x}_m)^{-1} - \tfrac{5}{24}(\tilde{x}_m)^{-3}.. \\ .. - \tfrac{83}{480}(\tilde{x}_m)^{-5} - \tfrac{2407}{13440}(\tilde{x}_m)^{-7} - ... \tag{17}$$

III.2.2. Algebraic approximation

By limiting this expansion, simple closed-form approximations are obtained for x_m. Table 4 shows the values obtained for x_0, x_1, x_2 using the successive terms of the series expansion; the table starts with $x_m \simeq \tilde{x}_m$ at rank $p = 0$ $(n = 2p-1)$. It can be observed for example that after only two ranks, the relative error of the very simple algebraic approximation $x_m \simeq \tilde{x}_m - \frac{1}{2}(\tilde{x}_m)^{-1} - \frac{5}{24}(\tilde{x}_m)^{-3}$ is less than 2.85x10^{-2}, 1.70x10^{-5} and 7.56x10^{-7}, for $m = 0$, 1 and 2 respectively. On the other hand, a relative error of less than 10^{-6} is obtained after 21, 3 and 2 ranks, for $m = 0$, 1 and 2 respectively. So, apart the case $m = 0$, the convergence is very fast. The relatively low convergence of the series expansion for $m = 0$ is mainly due to the fact that $\tilde{x}_0 = \pi/2$ presents a value not much higher than 1, unlike $\tilde{x}_m$ for $m > 0$. It can be observed that very simple algebraic approximations offer accurate results for x_m, except to some extend for $m = 0$.

Table 4. Successive values for x_0, x_1, x_2

p	x_0	x_1	x_2
0	1.570796327	4.712388980	7.853981634
1	1.252486441	4.606285685	7.790319657
2	1.198733883	4.604294850	7.789889636
3	1.180652258	4.604220440	7.789883850
4	1.173062329	4.604216969	7.789883753
∞	1.165561185	4.604216777	7.789883751

III.3. Amplitudes of the extrema

III.3.1. Series expansion

Using the same approach, the extrema themselves can be directly calculated. From eqn 12, their amplitudes are $y_m = 4x_m/(1+4x_m^2)$. Assuming that y_m can be expanded under the form $y_m = \sum_{n=0}^{+\infty} y_m^n (\tilde{x}_m)^{-n}$, and changing $\tilde{x}_m$ into z_m, eqns 14 and 15 lead to the simple following equation (m is omitted as above)

$$(1 - \frac{de}{dz})y + 2(z - e)\frac{de}{dz} = 0 \qquad (18)$$

Interpreting eqn 18 by using basic z-Transform properties allows here the straightforward computation of the expansion coefficients of y_m through the following relation (where n in y^n or e^n denotes an upper index)

$$y^n = 2ne^n - \sum_{j=0}^{n-1} (n-1-j)e^{n-1-j}(y^j + 2e^j) \qquad (19)$$

if n odd with $y_m^1 = 1$, and $y_m^n = 0$, if n even. Therefore the expansion of y_m begins as follows

$$y_m = (\tilde{x}_m)^{-1} + \tfrac{1}{4}(\tilde{x}_m)^{-3} + \tfrac{7}{48}(\tilde{x}_m)^{-5}.. \\ .. + \tfrac{37}{320}(\tilde{x}_m)^{-7} + \tfrac{8723}{80640}(\tilde{x}_m)^{-9} + ... \qquad (20)$$

III.3.2. Algebraic approximation

By limiting this expansion, simple closed-form approximations are obtained for y_m. Table 5 shows the values obtained for y_0, y_1, y_2 using the successive terms of the series expansion; the table starts with $y_m \simeq (\tilde{x}_m)^{-1}$ at rank $p = 0$ ($n = 2p + 1$). It can be observed for example that after only two ranks, the maximum relative error of the simple algebraic approximation $y_m \simeq (\tilde{x}_m)^{-1} + \frac{1}{4}(\tilde{x}_m)^{-3} + \frac{7}{48}(\tilde{x}_m)^{-5}$ is less than 1.14x10^{-2}, 1.09x$10^{-5}$and 4.99x10^{-7}, for $m = 0$, 1 and 2 respectively. On the other hand, a relative error of less than 10^{-6} is obtained after 17, 3 and 2 ranks, for $m = 0$, 1 and 2 respectively. So, apart the case $m = 0$, the convergence is very fast. The relatively low convergence of the series expansion for $m = 0$ is mainly due to the fact that $\widetilde{x}_0 = \pi/2$ presents a value not much higher than 1, unlike $\widetilde{x}_m$ for $m > 0$. It can be observed that very simple algebraic approximations offer accurate results for y_m, except to some extend for $m = 0$.

Table 5. Successive values for y_0, y_1, y_2

p	y_0	y_1	y_2
0	0.6366197724	0.2122065908	0.1273239545
1	0.7011228412	0.2145955933	0.1278399790
2	0.7163724049	0.2146583487	0.1278448589
3	0.7212725914	0.2146605893	0.1278449216
4	0.7231305518	0.2146606837	0.1278449226
∞	0.7246113538	0.2146606884	0.1278449226

IV. CONCLUSION

Methods for the computation of the extrema of the function $(\sin x/x)^r$ and the functionn $(\sin^2 x)/x$ have been presented. In both cases, the extrema locations and amplitudes are expressed under the form of series expansions. By exploiting Z-transform techniques, it is shown that the expansions coefficients can be obtained through straightforward recursive relations which can be easily implemented on a computer or a DSP. Very simple algebraic expressions are derived which give accurate values of the extrema locations and amplitudes, using only a very few coefficients - except, as far as $(\sin^2 x)/x$ is concerned, in the case of the first extremum.

For $(\sin x/x)^r$ as well as for $(\sin^2 x)/x$, both methods (recursive relations and algebraic expressions) can be useful for various applications in different domains such as signal processing and communications, but also in other fields of physics or engineering.

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