MODIFIED SOLUTION OF SOMMERFELD'S PROBLEM

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Abstract

The new rigorous boundary method based on solving Sommerfeld's problem for isotropic media is provided. Scattering of electromagnetic waves radiated by electrical point dipole near the plane surface boundary is considered in this paper. The problem is divided into two independent problems. Solution of each problem is investigated in the form of superposition of plane waves and reduced to the solution of system of algebraic equations. The solution of the boundary problem is represented in integral form for the general case. It coincides with the known solution by Sommerfeld, which is obtained from the solution of an ordinary differential equation. The new method may be used also in boundary problems for anisotropic media, which case will be investigated by our research group in the near future.

1. INTRODUCTION

Solving boundary problems is one of the actual problems of modern electrodynamics. The rigorous solving of the simplest key problem was considered by Sommerfeld in wave diffraction theory for the first time. This problem deals with the elementary dipole and with planar interface between two isotropic media. Helmholtz's equation was reduced to solving the ordinary differential equations and the rigorous solution was presented in the integral form by means of vector potentials by Sommerfeld [1]. This boundary problem (method of its solution and the obtained integral) is traditionally named the Sommerfeld problem in diffraction theory. It was considered in the work too [2] where the solution is derived with the help of Hertz vectors in the same way. Asymptotic analysis of Sommerfeld solution was researched by Malyuzhinets [3]. Many various methods such as the method of equivalent boundary conditions, the image method, Green's function method were developed to solve boundary problem on the basis of Sommerfeld problem. The method of equivalent boundary conditions is one of the effective methods for simplifying of boundary problems solving. For example, impedance boundary conditions of Schukin-Leontovich are applied for the problem of anisotropic media too [4]. The image method was first presented by Wait [5] and later by several other authors. For example, the rigorous solutions are obtained for cases of anisotropic half-space with perfectly electrically or magnetically conducting surface [6], for anisotropic half-space bounded by an anisotropic surface [7] and for a similarly anisotropic half-space and boundary [8] on the basis of the method of images. But there can be difficulties in determination of a image's form and its location. In the work [9] perfect conductor case with mixed-partial derivative boundary condition is considered on the basis of Green's function method for an anisotropic half-space.

In this work the modified new method of rigorously solving boundary problem is proposed on the basis of Sommerfeld key problem. The according to the proposed method the boundary problem is reduced to solving of system of algebraical equations concerning of Fourier component of density of surface current due to using the boundary conditions. The total electromagnetic field is determined through density of surface current induced on the interface. Thus the convenience and simplicity of calculation and the deep physical transparency are the main property of presented method. The method can be used in rigorously solving more difficult problems of electrodynamics particularly for cases of anisotropic media.

2. STATEMENT OF THE BOUNDARY PROBLEM

The system of Maxwell equations for stationary electromagnetic field is considered:

$$\begin{cases} \operatorname{rot} \boldsymbol{E} + i\omega\mu_0\mu\boldsymbol{H} = 0, \\ \operatorname{rot} \boldsymbol{H} - i\omega\varepsilon_0\varepsilon\boldsymbol{E} = \boldsymbol{j}, \end{cases} \qquad (1)$$

where $\boldsymbol{H}, \boldsymbol{E}$ are intensities of magnetic and electric fields, ε and μ are dielectric and magnetic permeability of medium, $\boldsymbol{j}$ is current density of external source. Time dependence is taken in the form of $e^{i\omega t}$.

Let the interface is the plane $x = 0$. Electric and magnetic permeability of the higher and lower half-spaces are characterized by the corresponding values: ε_1, μ_1 и ε_2, μ_2. For external source we choose elementary Hertz radiator located in a point along axis $x = x_0$ of the higher half-space and with the dipole momentum $\boldsymbol{p}$ directed along the axis *x*.

It is required to satisfy the electrodynamics boundary conditions on the planar interface (x=0), i.e. continuity of the tangential components of intensities.

It is necessary to determine electromagnetic field in any point of the space.

3. THE METHOD OF SOLVING BOUNDARY PROBLEM

3.1. General solution of system of Maxwell equations

Let's present the general solution of Maxwell's equations for isotropic media:

$$\boldsymbol{H} = -i\,\mathrm{F}^{-1}\left[\tilde{\psi}\left[\boldsymbol{k}, \tilde{\boldsymbol{J}}\right]\right],$$

$$\boldsymbol{E} = -\frac{i}{\omega\varepsilon\varepsilon_0}\mathrm{F}^{-1}\left[\tilde{\psi}\left(\varepsilon\mu k_0^2\tilde{\boldsymbol{J}} - \left(\boldsymbol{k}, \tilde{\boldsymbol{J}}\right)\boldsymbol{k}\right)\right], \qquad (2)$$

where F^{-1} is operator of inverse Fourier transformations, $\tilde{\boldsymbol{J}}$ is Fourier component of density of surface current, $\tilde{\psi} = \left(\varepsilon\mu k_0^2 - k_\rho^2 - k_x^2\right)^{-1}$, *k* is wave vector, *r* is radius vector.

Since the dipole moment is directed along the normal vector of the planar surface. It is obvious that problem is axisymmetric. Therefore it is convenient to consider the problem solution (2) in cylindrical frame:

$$\boldsymbol{H} = -\frac{i\boldsymbol{e}_\alpha}{(2\pi)^3}\int_0^\infty\int_0^{2\pi}\int_{-\infty}^{\infty} k_x \tilde{J}(k_\rho)\tilde{\psi}k_\rho \exp(i\boldsymbol{kr})\, dk_\rho d\alpha dk_x ,$$

$$\boldsymbol{E} = -\frac{i}{(2\pi)^3\varepsilon\varepsilon_0\omega}\int_0^\infty\int_0^{2\pi}\int_{-\infty}^{\infty}\left(\varepsilon\mu k_0^2\boldsymbol{e}_\rho - \boldsymbol{k}k_\rho\right)\tilde{J}(k_\rho)\tilde{\psi}k_\rho \exp(i\boldsymbol{kr})dk_\rho d\alpha dk_x . \qquad (3)$$

Using the following known identities for Bessel functions:

$$\frac{1}{2\pi}\int_0^{2\pi}\exp\left(ik_\rho\rho\cos\alpha\right)d\alpha = \mathrm{J}_0(k_\rho\rho),$$

$$\int_0^\infty \mathrm{J}_0\left(k_\rho\rho\right)dk_\rho = \frac{1}{2}\int_{-\infty}^{\infty}\mathrm{H}_0^{(1)}(k_\rho\rho)dk_\rho ,$$

and taking into account the scalar product $\boldsymbol{kr} = k_\rho\rho\cos(\alpha-\beta) + k_x x$ (*β* is angle between unit vectors e_ρ and e_z), the general solution of Maxwell's equations (3) can be put into the form:

$$\boldsymbol{H} = -\frac{i\boldsymbol{e}_\alpha}{8\pi^2}\iint_{R^2} k_x\tilde{J}(k_\rho)\tilde{\psi}k_\rho\,\mathrm{H}_0^{(1)}(k_\rho,\rho)\, dk_\rho dk_x ,$$

$$\boldsymbol{E} = -\frac{i}{8\pi^2\varepsilon\varepsilon_0\omega}\iint_{R^2}\left(\varepsilon\mu k_0^2\boldsymbol{e}_\rho - \boldsymbol{k}k_\rho\right)\tilde{J}(k_\rho)\tilde{\psi}k_\rho\,\mathrm{H}_0^{(1)}(k_\rho,\rho)dk_\rho dk_x , \qquad (4)$$

where $\mathrm{J}_0(x)$ - Bessel function for the zeroth order, $\mathrm{H}_0^{(1)}(x)$ - Henkel function of the first kind of the zeroth order.

3.2. Solving the boundary problem

Using the above relations in (4), refracted and reflected fields are written accordingly as:

$$\boldsymbol{H}^{\boldsymbol{R}} = -\frac{i\boldsymbol{e}_\alpha}{8\pi^2}\iint_{R^2} k_x\tilde{J}_1(k_\rho)\tilde{\psi}_1 k_\rho\,\mathrm{H}_0^{(1)}(k_\rho,\rho)\, dk_\rho dk_x ,$$

$$\boldsymbol{E}^{\boldsymbol{R}} = -\frac{i}{8\pi^2\varepsilon_1\varepsilon_0\omega}\iint_{R^2}\left(k_{01}^2\boldsymbol{e}_\rho - \boldsymbol{k}k_\rho\right)\tilde{J}_1(k_\rho)\tilde{\psi}_1 k_\rho\,\mathrm{H}_0^{(1)}(k_\rho,\rho)dk_\rho dk_x ,$$

$$\mathrm{H}^{\mathrm{T}} = -\frac{i\mathrm{e}_\alpha}{8\pi^2}\iint_{R^2} k_x\tilde{J}_2(k_\rho)\tilde{\psi}_2 k_\rho\,\mathrm{H}_0^{(1)}(k_\rho,\rho)\, dk_\rho dk_x ,$$

$$\boldsymbol{E}^{\boldsymbol{T}} = -\frac{i}{8\pi^2\varepsilon_2\varepsilon_0\omega}\iint_{R^2}\left(k_{02}^2\boldsymbol{e}_\rho - \boldsymbol{k}k_\rho\right)\tilde{J}_2(k_\rho)\tilde{\psi}_2 k_\rho\,\mathrm{H}_0^{(1)}(k_\rho,\rho)dk_\rho dk_x , \qquad (5)$$

where $k_{01} = \sqrt{\varepsilon_1 \mu_1} k_0$, $k_{02} = \sqrt{\varepsilon_2 \mu_2} k_0$ - wave numbers in the higher and lower half-spaces, $\tilde{\psi}_1 = \dfrac{1}{k_{01}^2 - k_\rho^2 - k_x^2}$, $\tilde{\psi}_2 = \dfrac{1}{k_{02}^2 - k_\rho^2 - k_x^2}$, $\tilde{\boldsymbol{J}}_1 = \tilde{J}_1(k_\rho)\boldsymbol{e}_\rho$, $\tilde{\boldsymbol{J}}_2 = \tilde{J}_2(k_\rho)\boldsymbol{e}_\rho$ - Fourier component of density of surface current. The field of point dipole is written as:

$$\boldsymbol{E}^0 = -(\varepsilon_1 \varepsilon_0)^{-1}\left(\mathrm{graddiv} + k_{01}^2\right)(\psi_1 \boldsymbol{p}),$$

$$\boldsymbol{H}^0 = i\omega \,\mathrm{rot}(\psi_1 \boldsymbol{p}). \quad (6)$$

Then total solution of electromagnetic field is presented in form:

$$\boldsymbol{H} = \begin{cases} \boldsymbol{H}^- = \boldsymbol{H}^0 + \boldsymbol{H}^R, x > 0; \\ \boldsymbol{H}^T, x < 0; \end{cases},$$

$$\boldsymbol{E} = \begin{cases} \boldsymbol{E}^- = \boldsymbol{E}^0 + \boldsymbol{E}^R, x > 0; \\ \boldsymbol{E}^T, x < 0. \end{cases}. \quad (7)$$

Calculating integral in (5) by k_x with help of the residue theory we obtain the following relations for $\boldsymbol{H}^-, \boldsymbol{E}^-, \boldsymbol{H}^T, \boldsymbol{E}^T$ in (7):

in the higher half-space:

$$\boldsymbol{H}^- = \boldsymbol{H}^0 - \frac{\boldsymbol{e}_\alpha}{8\pi} \int_{-\infty}^{\infty} \tilde{J}_1(k_\rho) e^{i\kappa_1 x}\, \mathrm{H}_0^{(1)}(k_\rho, \rho)\, k_\rho dk_\rho,$$

$$\boldsymbol{E}^- = \boldsymbol{E}^0 - \frac{1}{8\pi^2 \varepsilon_1 \varepsilon_0 \omega} \int_{-\infty}^{\infty} (\boldsymbol{e}_\rho \kappa_1 - \boldsymbol{e}_x) \tilde{J}_1(k_\rho)\, e^{ik_1 x} k_\rho\, \mathrm{H}_0^{(1)}(k_\rho, \rho), \quad (8)$$

in the lower half-space:

$$\boldsymbol{H}^T = \frac{\boldsymbol{e}_\alpha}{8\pi} \int_{-\infty}^{\infty} \tilde{J}_2(k_\rho) e^{-i\kappa_2 x} k_\rho\, \mathrm{H}_0^{(1)}(k_\rho, \rho)\, dk_\rho,$$

$$\boldsymbol{E}^T = -\frac{1}{8\pi^2 \varepsilon_2 \varepsilon_0 \omega} \int_{-\infty}^{\infty} (\boldsymbol{e}_\rho \kappa_2 + \boldsymbol{e}_x) \tilde{J}_2(k_\rho)\, e^{-ik_2 x} k_\rho\, \mathrm{H}_0^{(1)}(k_\rho, \rho), \quad (9)$$

where $\kappa_1 = \sqrt{k_{01}^2 - k_\rho^2}$, $\kappa_2 = \sqrt{k_{02}^2 - k_\rho^2}$.

3.3. Boundary conditions

Taking into account nonzero single tangential component of electromagnetic field in (8), (9) we require satisfaction of boundary condition in form ($x = 0$):

$$H_\alpha^o + H_\alpha^R = H_\alpha^T, \quad E_\rho^o + E_\rho^R = E_\rho^T, \quad (10)$$

where

$$H_\alpha^0 = -\frac{1}{8\pi} \int_{-\infty}^{\infty} \frac{i\omega p k_\rho e^{i\kappa_1 x_0}}{\kappa_1} \mathrm{H}_0^{(1)}(k_\rho, \rho) k_\rho dk_\rho,$$

$$E_\rho^0 = \frac{i}{8\pi \varepsilon_1 \varepsilon_0} \int_{-\infty}^{\infty} \omega p k_\rho e^{i\kappa_1 x_0}\, \mathrm{H}_0^{(1)}(k_\rho, \rho) k_\rho dk_\rho,$$

$$H_\alpha^R = -\frac{1}{8\pi} \int_{-\infty}^{\infty} \tilde{J}_1(k_\rho) k_\rho\, \mathrm{H}_0^{(1)}(k_\rho, \rho) dk_\rho,$$

$$E_\rho^R = -\frac{1}{8\pi^2 \varepsilon_1 \varepsilon_0 \omega} \int_{-\infty}^{\infty} \kappa_1 \tilde{J}_1(k_\rho) k_\rho\, \mathrm{H}_0^{(1)}(k_\rho, \rho)\, dk_\rho,$$

$$H_\alpha^T = \frac{1}{8\pi} \int_0^{\infty} \int_0^{2\pi} \tilde{J}_2(k_\rho) k_\rho\, \mathrm{H}_0^{(1)}(k_\rho, \rho) dk_\rho,$$

$$E_\rho^T = \frac{-1}{8\pi^2 \varepsilon_2 \varepsilon_0 \omega} \int_{-\infty}^{\infty} \kappa_2 \tilde{J}_2(k_\rho) k_\rho\, \mathrm{H}_0^{(1)}(k_\rho, \rho) dk_\rho.$$

Then from (10) we derive:

$$\frac{1}{8\pi} \int_{-\infty}^{\infty} \left(\frac{i\omega p k_\rho e^{i\kappa_1 x_0}}{\kappa_1} + \tilde{J}_1(k_\rho) \right) \mathrm{H}_0^{(1)}(k_\rho, \rho)\, k_\rho dk_\rho = -\frac{1}{8\pi} \int_{-\infty}^{\infty} \tilde{J}_2(k_\rho) \mathrm{H}_0^{(1)}(k_\rho, \rho) k_\rho dk_\rho,$$

$$\frac{1}{8\pi \varepsilon_1 \varepsilon_0} \int_{-\infty}^{\infty} \left(-i\omega p k_\rho e^{i\kappa_1 x_0} + \tilde{J}_1(k_\rho)\kappa_1 \right) \mathrm{H}_0^{(1)}(k_\rho, \rho) k_\rho dk_\rho = \frac{1}{8\pi \varepsilon_2 \varepsilon_0} \int_{-\infty}^{\infty} \tilde{J}_2(k_\rho)\kappa_2\, \mathrm{H}_0^{(1)}(k_\rho, \rho) k_\rho dk_\rho. \quad (11)$$

From (11) we obtain the following system of algebraical equations:

$$\begin{cases} \dfrac{i\omega p k_\rho e^{i\kappa_1 x_0}}{\kappa_1} + \tilde{J}_1(k_\rho) = -\tilde{J}_2(k_\rho), \\ -i\omega p k_\rho e^{i\kappa_1 x_0} + \tilde{J}_1(k_\rho)\kappa_1 = \dfrac{\varepsilon_1}{\varepsilon_2} \tilde{J}_2(k_\rho)\kappa_2. \end{cases} \quad (12)$$

The solution of system (12) are Fourier component of sought surface current density:

$$\tilde{J}_1(k_\rho) = i\omega p k_\rho e^{i\kappa_1 x_0} \frac{\varepsilon_2 \kappa_1 - \varepsilon_1 \kappa_2}{\kappa_1(\varepsilon_2 \kappa_1 + \varepsilon_1 \kappa_2)},$$

$$\widetilde{J}_2(k_\rho) = -i\omega p k_\rho e^{i\kappa_1 x_0} \frac{2\varepsilon_2}{\varepsilon_2\kappa_1 + \varepsilon_1\kappa_2}. \qquad (13)$$

3.4. Total solution of boundary problem

Substituting those Fourier components of surface current densities (13) in (8), (9) we derive general solution of problem in form:

in the higher half-space:

$$\boldsymbol{H^T} = -\frac{i\omega p \boldsymbol{e}_\alpha}{4\pi} \int_{-\infty}^{\infty} \frac{\varepsilon_2 e^{i(\kappa_1 x_0 - \kappa_2 x)}}{\varepsilon_2\kappa_1 + \varepsilon_1\kappa_2} \mathrm{H}_0^{(1)}(k_\rho \rho)$$

$$k_\rho^2 dk_\rho,$$

$$\boldsymbol{E^T} = -\frac{ip}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} (\boldsymbol{e}_x \frac{1}{\kappa_2} - \boldsymbol{e}_\rho) \frac{\kappa_2 e^{i(\kappa_1 x_0 - \kappa_2 x)}}{\varepsilon_2\kappa_1 + \varepsilon_1\kappa_2} \qquad (14)$$

$$k_\rho^2 \mathrm{H}_0^{(1)}(k_\rho, \rho) dk_\rho,$$

in the lower half-space:

$$\boldsymbol{H}^- = \boldsymbol{H^0} - \frac{i\omega p \boldsymbol{e}_\alpha}{8\pi} \int_{-\infty}^{\infty} \frac{\varepsilon_2\kappa_1 - \varepsilon_1\kappa_2}{\kappa_1(\varepsilon_2\kappa_1 + \varepsilon_1\kappa_2)} k_\rho^2$$

$$\mathrm{H}_0^{(1)}(k_\rho \rho) e^{i\kappa_1(x_0 + x)} dk_\rho,$$

$$\boldsymbol{E}^- = \boldsymbol{E^0} - \frac{ip}{8\pi\varepsilon_0} \int_{-\infty}^{\infty} (\boldsymbol{e}_x \frac{1}{\kappa_1} + \boldsymbol{e}_\rho) e^{i\kappa_1(x + x_0)}$$

$$\frac{\varepsilon_2\kappa_1 - \varepsilon_1\kappa_2}{\varepsilon_1(\varepsilon_2\kappa_1 + \varepsilon_1\kappa_2)} k_\rho^2 \mathrm{H}_0^{(1)}(k_\rho, \rho) dk_\rho. \qquad (15)$$

4. CONCLUSION

Thus the new rigorous method for solving boundary problem is presented on the basis of Sommerfeld key problem. The physical meaning of the method consists in definition of surface current density by using the Fourier transformations and Green's functions.

The proposed method can be applied in various boundary problems for anisotropic media.

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