# UNSYMMETRICAL ELECTROMAGNETIC WAVE DIFFRACTION BY A LONG PIPE

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***Abstract***

*Diffraction of an unsymmetrical electromagnetic wave by a long pipe coaxially oriented inside an infinite waveguide is considered. The corresponding boundary value problem is reduced to a system of singular integral equations concerning the Fourier component of the surface current density. The exact solution of the above system of equations is constructed by the Wiener-Hopf-Fok method in a class of analytical functions and it is defined in the form of sum of partial waves.*

## 1. Introduction

Unsymmetrical $E_{nm}$ (electrical or TM) and $H_{nm}$ (magnetic or TE) waves ($m$ = 1, 2, 3, …) differ from symmetrical waves ($m$ = 0) that the diffraction field of unsymmetrical waves is characterized by two scalar functions which correspond to a longitudinal component of electric and magnetic Hertz's vectors according to the following equations [1]:

$$\Pi_z^e = \sin(m\varphi + \varphi_0)\Pi(r,z), \quad \Pi_z^m = \cos(m\varphi + \varphi_0)\tilde{\Pi}(r,z).$$

The presence of the two Hertz potentials complicates the derivation of the boundary value problem by Wiener-Hopf-Fok method [2, 3]. However the exact solution of this problem can be obtained by some generalization of the corresponding axially symmetrical problem.

Electromagnetic fields are expressed in terms of the functions $\Pi$ and $\tilde{\Pi}$ as follows:

$$E_r = \sin(m\varphi + \varphi_0)\left(\frac{\partial^2}{\partial r \partial z}\Pi - i\frac{mk}{r}W\tilde{\Pi}\right),$$

$$E_\varphi = \cos(m\varphi + \varphi_0)\left(\frac{m}{r}\frac{\partial}{\partial z}\Pi - ikW\frac{\partial}{\partial r}\tilde{\Pi}\right),$$

$$E_z = \sin(m\varphi + \varphi_0)\left(\frac{\partial^2}{\partial z^2} + k^2\right)\Pi,$$

$$H_r = \cos(m\varphi + \varphi_0)\left(\frac{\partial^2}{\partial r \partial z}\tilde{\Pi} - i\frac{mk}{r}W^{-1}\Pi\right),$$

$$H_\varphi = -\sin(m\varphi + \varphi_0)\left(\frac{m}{r}\frac{\partial}{\partial z}\tilde{\Pi} - ikW^{-1}\frac{\partial}{\partial r}\Pi\right),$$

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$$H_z = \cos(m\varphi + \varphi_0)\left(\frac{\partial^2}{\partial z^2} + k^2\right)\tilde{\Pi}, \quad W = \sqrt{\frac{\mu}{\varepsilon}}, \quad k = \frac{\omega}{c}.$$

The constant angle $\varphi_0$ is determined by polarization of the wave impinging on the end of the circular waveguide.

Note that electromagnetic field of $E_{mn}$ waves is defined by electric Hertz function $\Pi$ from the abovementioned formulas, and electromagnetic field of waves $H_{mn}$ by magnetic Hertz function $\tilde{\Pi}$.

Thus it is necessary to consider jointly unsymmetrical waves $E_{m1}$, $E_{m2}$, ... and $H_{m1}$, $H_{m2}$, ... for the given value *m* (*m*=1, 2, 3 ...), as they are transformed each other at reflection from the end of the waveguide.

## 2. Statement of the problem

Let two waves are incident from the right to the left at the end of the long pipe with infinitely thin wall of radius $a_1$ located coaxially in the basic waveguide of radius *a*: one is unsymmetrical TM-wave with amplitude *A* and wave number *h* and the other is unsymmetrical TE-wave with amplitude *B* and wave number $\tilde{h}$ (fig. 1).

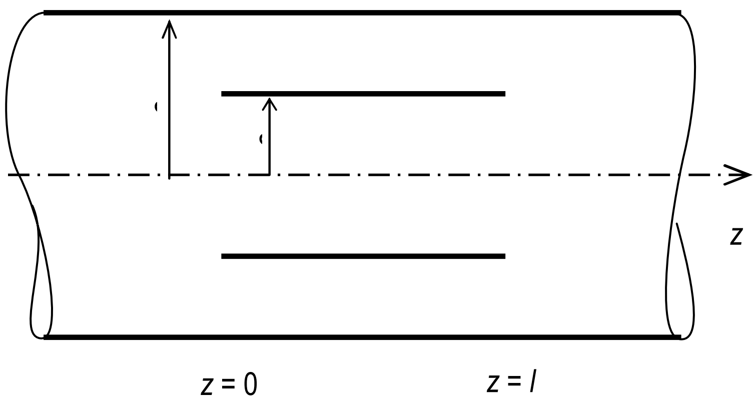


**Fig. 1**

The problem solution should satisfy the following boundary conditions:

$$E_z = E_\varphi = 0 \text{ at } r = a, \ -\infty < z < \infty; \ r=a_1, \ 0 \le z \le l, \tag{1}$$

$$J_\varphi = H_z(a_1 - 0, \varphi, z) - H_z(a_1 + 0, \varphi, z) = 0 \text{ at } z < 0, \ z > l, \tag{2}$$

$$J_z = H_\varphi(a_1 + 0, \varphi, z) - H_\varphi(a_1 - 0, \varphi, z) = 0 \text{ at } z \le 0, \ z \ge l, \tag{3}$$

where $J_\varphi$, $J_z$ are azimuthally and longitudinal components of the surface current density.

The electrical and magnetic Hertz functions $\Pi$ and $\tilde{\Pi}$ should be the solutions of the equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\Pi\right) + \frac{\partial^2}{\partial z^2}\Pi + \left(k^2 - \frac{m^2}{r^2}\right)\Pi = 0. \tag{4}$$

## 3. Solution of the problem

We are looking for Hertz functions according to equation (4) in the following form [1]:

$$\Pi = -i\frac{\pi a_1}{2k} W \int_C \exp(iwz)\,\mathrm{L}(r,w)\,\mathrm{F}(w)dw, \tag{5}$$

$$\tilde{\Pi} = \frac{\pi a_1}{2} \int_C \exp(iwz) L(r,w) \frac{F(w)}{\mathrm{v}} dw, \tag{6}$$

where $\mathrm{v} = \sqrt{k^2 - w^2}$ ,

$$\mathrm{L}(r,w) = \frac{1}{\mathrm{J}_m(\mathrm{v}a)} \begin{cases} \mathrm{J}_m(\mathrm{v}r)(a_1,a),\ r \le a_1 \\ \mathrm{J}_m(\mathrm{v}a_1)(r,a),\ r \ge a_1 \end{cases},$$

$$L(r,w) = \frac{1}{\mathrm{J}'_m(\mathrm{v}a)} \begin{cases} \mathrm{J}_m(\mathrm{v}r)(a'_1,a'),\ r \le a_1 \\ \mathrm{J}'_m(\mathrm{v}a_1)(r,a'),\ r \ge a_1 \end{cases},$$

$$(r,a) = \mathrm{J}_m(\mathrm{v}r)\,\mathrm{N}_m(\mathrm{v}a) - \mathrm{J}_m(\mathrm{v}a)\,\mathrm{N}_m(\mathrm{v}r),$$

$$(r,a') = \mathrm{J}_m(\mathrm{v}r)\,\mathrm{N}'_m(\mathrm{v}a) - \mathrm{J}'_m(\mathrm{v}a)\,\mathrm{N}_m(\mathrm{v}r),$$

$$(r',a') = \mathrm{J}'_m(\mathrm{v}r)\,\mathrm{N}'_m(\mathrm{v}a) - \mathrm{J}'_m(\mathrm{v}a)\,\mathrm{N}'_m(\mathrm{v}r),$$

$\mathrm{J}_m(x)$ is the Bessel function, $\mathrm{N}_m(x)$ is the Neumann function, *C* is the integration contour in the complex plane *w* lying along the real axis and consisting of an infinitely narrow loop enveloping a point $h$ and $\tilde{h}$ from below, F and *F* are the decision functions.

The boundary value problem is reduced with the help of the boundary conditions (1) – (3) to the system of the following functional integral equations:

$$\int_C \exp(iwz)\left( i\frac{mw}{k^2 a_1}\mathrm{L}\mathrm{F}(w) + LF(w) \right) dw = 0 \quad \text{at } 0 \le z \le l, \tag{7}$$

$$\int_C \exp(iwz)\,\mathrm{v}^2\,\mathrm{L}\mathrm{F}(w)dw = 0 \quad \text{at } 0 \le z \le l, \tag{8}$$

$$\int_C \exp(iwz) F(w) dw = 0 \quad \text{at } z < 0,\ z > l \tag{9}$$

$$\int_C \exp(iwz)\left( \mathrm{F}(w) + i\frac{mw}{a_1 \mathrm{v}^2} F(w) \right) dw = 0 \quad \text{at } z \le 0,\ z \ge l, \tag{10}$$

where the following notation is introduced : $\mathrm{L} \equiv \mathrm{L}(a_1, w)$, $L \equiv L(a'_1, w)$.

Taking into account that the edges of the pipe are secondary sources of waves, the Fourier-component of the current density is constructed by Wiener-Hopf-Fok method as a sum from two analytical sources in the form of natural space harmonics forward and backward:

$$\mathrm{F}(w) = \frac{1}{(k-w)\mathrm{L}_-}\left( C_1^+(w) + \frac{C_1}{w+k} \right) + \frac{\exp(-iwl)}{(k+w)\mathrm{L}_+}\left( C_2^-(w) + \frac{C_2}{w-k} + \frac{D_2}{w+h} \right), \tag{11}$$

$$F(w)=\frac{1}{L_-}\left(\frac{B_1}{w+k}+\frac{A_1}{w-k}+E_1^+(w)\right)+$$
$$+\frac{\exp(-iwl)}{L_+}\left(\frac{B_2}{w+k}+\frac{A_2}{w-k}+E_2^-(w)+\frac{F_2}{w+\tilde{h}}\right), \tag{12}$$

where $C_1$, $C_2$, $D_2$, $A_1$, $A_2$, $B_1$, $B_2$, $F_2$ are constants, $C_1^+(w)$, $E_1^+(w)$ are analytical functions on the upper complex *w* plane, $C_2^-(w)$, $E_2^-(w)$ are analytical functions on the lower complex plane.

As the integral along an infinitely narrow loop of the contour *C* corresponds to amplitude of the incident wave, it is easy to calculate the values of the following constants:

$$D_2=A\frac{k}{\pi^2 a_1}\frac{\mathrm{L}_-(a_1,h)}{(k+h)}\exp(-ihl)\left.\frac{\mathrm{J}_m(\mathrm{v}a)}{(a_1,a)}\right|_{\mathrm{v}=\sqrt{k^2-h^2}}, \tag{13}$$

$$F_2=-i\frac{B}{\pi^2 a_1}L_-(a_1,\tilde{h})\exp(-i\tilde{h}l)\left.\frac{\mathrm{J}'_m(\mathrm{v}a)}{(a'_1,a')}\right|_{\mathrm{v}=\sqrt{k^2-\tilde{h}^2}}. \tag{14}$$

Similarly we have

$$B_1=-B_2\frac{L_+(a_1,k)}{L_-(a_1,k)}\exp(ikl),$$

$$A_2=-A_1\frac{L_+(a_1,k)}{L_-(a_1,k)}\exp(ikl).$$

By substituting expressions (11), (12) into the system of the integral equations (7) - (10) and closing the integration contour *C* in the upper half-plane or in the lower half-plane along the infinite semicircle according to Jordan's lemma, it is easy to obtain the system of the linear algebraic and functional equations, as following:

$$\frac{C_1}{2k\,\mathrm{L}_+(a_1,k)}+\frac{1}{\mathrm{L}_-(a_1,k)}\left(C_2^-(-k)-\frac{C_2}{2k}+\frac{D_2}{h-k}\right)\exp(ikl)=$$
$$=\frac{im}{2a_1}\left[\frac{1}{L_+(a_1,k)}\left(E_1^+(-k)-\frac{A_1}{2k}\right)+\frac{1}{L_-(a_1,k)}\left(E_2^-(-k)-\frac{A_2}{2k}+\frac{F_2}{\tilde{h}-k}\right)\exp(ikl)\right], \tag{15}$$

$$\frac{1}{\mathrm{L}_-(a_1,k)}\left(\frac{C_1}{2k}+C_1^+(k)\right)+\frac{C_2}{2k\,\mathrm{L}_+(a_1,k)}\exp(-ikl)=$$
$$=\frac{im}{2a_1}\left[\frac{1}{L_-(a_1,k)}\left(E_1^+(k)+\frac{B_1}{2k}\right)+\frac{1}{L_+(a_1,k)}\left(E_2^-(k)+\frac{B_2}{2k}+\frac{F_2}{\tilde{h}+k}\right)\exp(-ikl)\right]. \tag{16}$$

$$\left(\frac{im}{a_1 k}\frac{L_-(a_1,k)}{2k}C_2+L_-(a_1,k)A_2\right)\exp(-ikl)+L_+(a_1,k)A_1=$$
$$=\frac{im}{a_1 k}\mathrm{L}_+(a_1,k)\left(\frac{C_1}{2k}+C_1^+(k)\right), \tag{17}$$

$$\frac{im}{a_1 k}\left( \mathrm{L}_-(a_1,k)\frac{C_1}{2k} + L_+(a_1,k)\left(\frac{C_2}{2k} - C_2^-(-k) + \frac{D_2}{k-h}\right)\exp(ikl)\right) = \\ = L_-(a_1,k)B_1 + L_+(a_1,k)B_2\exp(ikl). \tag{18}$$

$$C_1^+(w) = -\sum_{n=1}^{\infty}\frac{\exp(iw_n l)}{w+w_n}\frac{\mathrm{L}_+(a_1,w_n)}{\mathrm{L}_-^*(a_1,w_n)}\frac{(k+w_n)}{(k-w_n)}\left( C_2^-(-w_n) - \frac{C_2}{k+w_n} + \frac{D_2}{h-w_n}\right), \tag{19}$$

$$C_2^-(w) = -\sum_{n=1}^{\infty}\frac{\exp(iw_n l)}{w-w_n}\frac{\mathrm{L}_+(a_1,w_n)}{\mathrm{L}_-^*(a_1,w_n)}\frac{(k+w_n)}{(k-w_n)}\left( C_1^+(w_n) + \frac{C_1}{k+w_n}\right), \tag{20}$$

where $w_n$ are the zeros of $\mathrm{L}_-$ ($n$ = 1, 2, …),

$$\mathrm{L}_-^*(a_1,w_n) = \lim_{w\to w_n}(w-w_n)^{-1}\,\mathrm{L}_-(a_1,w).$$

It is necessary to note, reasonably, the convergence of the infinite series, on account of exponential convergence, and in consideration of all traveling spatial harmonics and of some damped harmonics with imaginary wave numbers. Thus, the boundary value problem was reduced to the solution of a finite system of linear algebraic equations.

## References

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[3] B. Noble, *Methods based on the Wiener-Hopf technique*, 2nd ed., New York, Chelsea, 1988.