2-D FILTERS SYNTHESIS USING METHOD OF COMPRESSED COSINES

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Abstract

The method of compressed cosines is used for mathematical approximations. With it, polynomials of a low order are obtained that approximate with high accuracy ideal functions with rectangular contours. Up till now it has been used for synthesis of digital filters for one-dimensional signals. This paper considers the application of the method for synthesis of two-dimensional digital filters. Filters are obtained with characteristics close to the ideal ones.

1. INTRODUCTION

In every technical device using information signals the purpose of filters is to separate the signals necessary for its functioning and to suppress all others that are interference to it. The ideal filter is a rectangular contour. This is a function of the frequency ω having two regions: passband (PB) – the function is equal to unity, and stopband (SB) – the function is equal to zero. It is defined by the expression

$$H_d(\omega) = \begin{cases} 1, \omega \in [0, \omega_c] \\ 0, \omega \in (\omega_c, \pi] \end{cases}. \qquad (1)$$

where ω_c is the cutoff frequency.

A filter with an ideal response cannot be realized. This is why the ideal function is approximated with another one that can be realized by technical means. The task is to define such an approximation function that is very close to the ideal one, and that has a low computational complexity. The approximating functions can be fractional rational, spline functions, polynomials, etc. For technical devices the fractional rational approximations have the best properties but in many cases they are not applicable. In such cases polynomial approximations are used.

A very important factor in these mathematical tasks is the criterion (metric) L_p, $p = 1 \div \infty$ used for approximating the ideal function. Fig. 1 shows a comparison among the most used metrics L_1, L_2 and L_∞ in approximations with polynomials of 32nd degree. It is seen that in filter synthesis a compromise has to be found between two contradictory requirements: the amplitude of the oscillations and the steepness of the function in the transition band – the band in which the function goes from unity to zero. In all the criteria the function has oscillations in the passband and the stopband that are proportional to the degree of the polynomial.

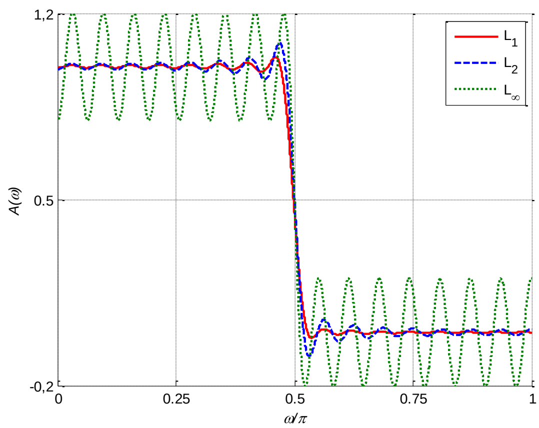

Fig.1. Approximations with polynomials of 32nd degree in an L_1, L_2 and L_∞ metric

These oscillations are undesirable. Their amplitude reflects the approximation error ε. The purpose in filter synthesis is to obtain the rectangular contour of the ideal function that is maximally flat characteristics in the passband and stopband, and the narrowest possible transition band. With L_1 and

L_2 metrics the oscillations increase near the transition band of the function. This is due to the Gibbs effect [1].

Numerous methods exist for reducing this effect. It can be generalized that every action towards decreasing the amplitude inevitably leads either to broadening the transition band or to increasing the degree of the polynomial which decreases the accuracy of the approximation and increases its computational complexity.

The approximations in an L_∞ metric are performed with the well-known method of Parks-McClellan [2], which is a modified version of the second algorithm of Remez [3]. It is a minimax, equiripple approximation relative to the Chebyshev distance. It is performed with a trigonometric polynomial of degree m:

$$A_m(x) = \sum_{n=0}^{m} b_n \cos(nx);\; x \in [0,1], \quad (2)$$

where b_n are the polynomial coefficients, $x = \omega/\pi$. With this method the transition band can be precisely defined and arbitrary narrow. The latter, however, leads to an increase of the approximation error, resp. the amplitude of the oscillations. An important advantage of the method is that the synthesis is performed by an iterative algorithm of Remez that has a fast convergence and a low computational complexity. It is established that with equal specification (non-uniformity in PB and SB and equal transition bandwidth) with the Parks-McClellan method the approximation is performed by a polynomial of the lowest degree. This determines the broad applicability of the method in many technical tasks.

2. ESSENCE OF THE METHOD OF COMPRESSED COSINES

In [4] a new approximation method called, "compressed cosines", using chebyshev's norm is proposed. With the method a low degree trigonometrical polynomials with small approximation error are determined. The analytical expression of the polynomial is

$$A_m(x) = \sum_{k=1}^{m+1} b_k \cos\left\{(k-1)\frac{-\pi}{2}\left[\operatorname{erf}\left(\beta(2x-1)\right)+1\right]\right\} =$$

$$= \sum_{k=1}^{m+1} b_k \cos\left[(k-1)\varphi(x)\right],\; x \in [0,1]. \quad (3)$$

It is seen that the argument of cosine consist modulating function

$$\varphi(x) = -\frac{\pi}{2}\left[\operatorname{erf}\left(\beta(2x-1)\right)+1\right], \quad (4)$$

The functoin $\operatorname{erf}(.)$ is the Gauss integral error function. It has S - shaped graph. The slope of the graph depends on the parameter β, and compresses the oscillations of cosine – fig. 2.

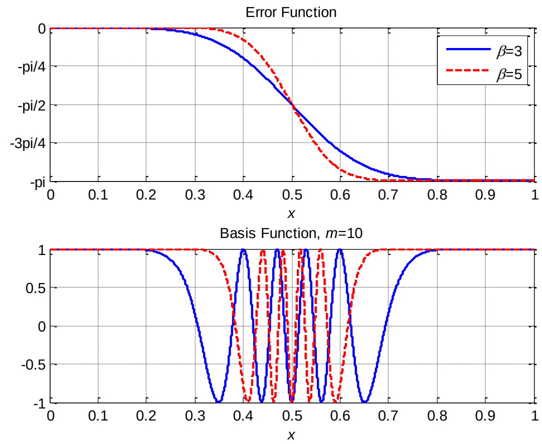

Fig. 2. Modulating function erf(.) and compression of cosine

Fig. 3 shows an optimal approximation by a polynomial of the lowest possible degree (fourth).

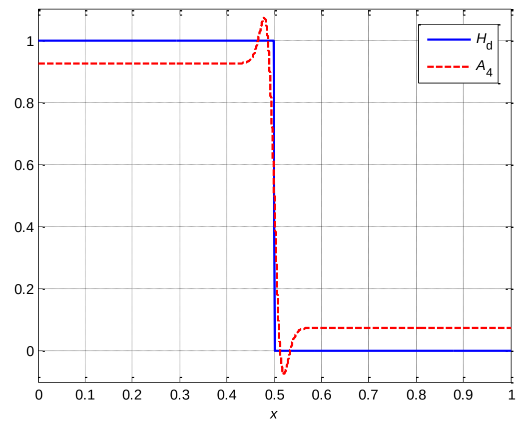

Fig. 3. Optimal approximation of fourth degree

It is seen from the figure that the approximation has only two extreme points, while with the other methods the extreme points are much more (Fig. 1), which is due to the low degree of the polynomial. In this regard this approximation is closest to the ideal function. The polynomial coefficients are obtained by the Remez algorithm that is an iterative solution to a system of $m+2=6$ linear equations. With the other two approximations the equations are much more. For filters with specifications close to the ideal function, the reduction of the computation operations is more than 100 times.

On the other hand, in the proposed method the magnitude of the "jump" of the function decreases with the increase of the parameter β, without change of the transition bandwidth or increasing the polynomial degree. Fig. 4 shows this valuable property of the method for a fixed transition band and two different values of the parameter β.

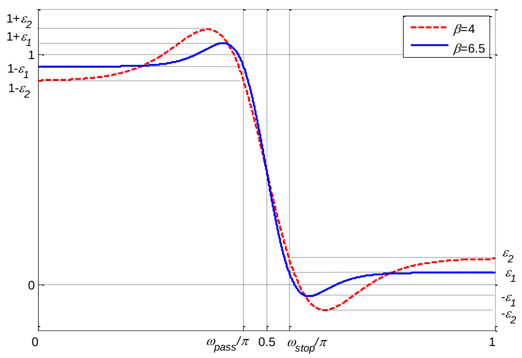

Fig. 4. Approximation error ε depending on the parameter β for a fixed transition band

The parameter β can be arbitrarily large and the transition band – arbitrarily narrow. With the proposed method approximations to ideal functions with a very high accuracy and a very low computational complexity can be done. With a transition bandwidth equal to zero and $\beta = \infty$ the fourth degree polynomial coincides with the ideal transfer function. Instructively speaking, the proposed polynomial of fourth degree is the shortest way to the ideal function.

3. APPLICATION OF THE METHOD TO COMPUTING 2D FILTERS

The magnitude response of one-dimensional digital filters is a function of one variable which designates the frequency. The general form of the transfer function is

$$H(x) = h_0 + \sum_n h_n \cos(nx), \qquad (5)$$

where h_n are the filter coefficients. The plot of the magnitude response (MR) is a plane figure (fig. 4). The abscissa is the argument and the ordinate is the amplitude value.

In the two-dimensional filters the transfer function is of two variables

$$H(x,y) \square \sum_{n_1} \sum_{n_2} 2h(n_1,n_2)\cos(n_1 x + n_2 y). \qquad (6)$$

The MR is a spatial figure whose base is the plane defined by the values of the two arguments. The values of the arguments are on the abscissa x and the ordinate y, and the amplitude is on the applicate z. The ideal transfer function of a lowpass two-dimensional filter is a rectangular contour whose rotational body is a cylinder.

Approximation in a two-dimensional space is a complicated and time-consuming operation [5]. This is why the fastest way is to perform a one-dimensional approximation and to obtain the spatial figure by a program as a rotational body with respect to the ordinate.

Fig. 5 shows an MR of a one-dimensional lowpass filter with a specification: degree of the approximating polynomial $m = 4$, attenuation in the stopband $DS \geq 10$ dB; normed transition frequency $f_c = 0.5$; normed transition bandwidth $\Delta f = 0.1$

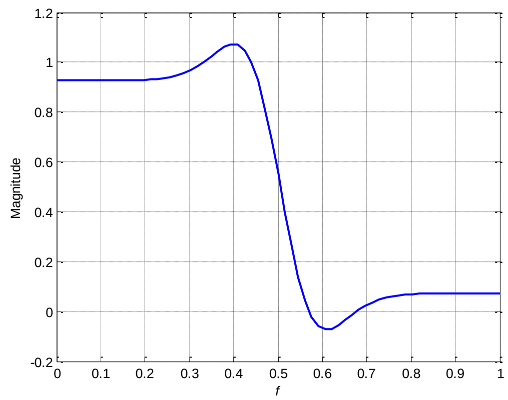

Fig. 5. MR of a one-dimensional LP filter

An MR of the derived two-dimensional filter is obtained by the rotation of the one-dimensional MR with respect to the ordinate Fig. 6.

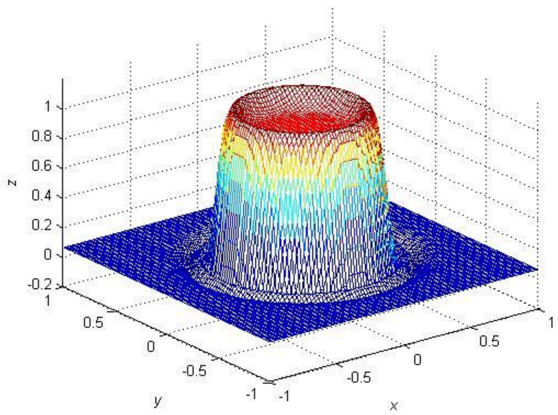

Fig. 6. Two-dimensional lowpass filter

In the example shown the filter specification is deliberately chosen to have a low selectivity in order to obtain a better visualization of the plots. Obviously, if the transition bandwidth is decreased and the value of the parameter β is increased, the filter with compressed cosines will have a characteristic closer to the ideal one – a cylinder.

In a similar way a two-dimensional highpass filter is obtained. Fig. 7 shows a highpass two-dimensional filter, mirroring the prototype from Fig. 6.

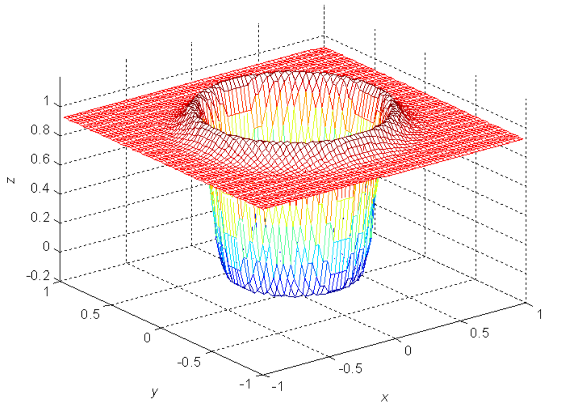

Fig. 7. Two-dimensional highpass filter

Fig. 8 show characteristics of a two-dimensional bandpass filter.

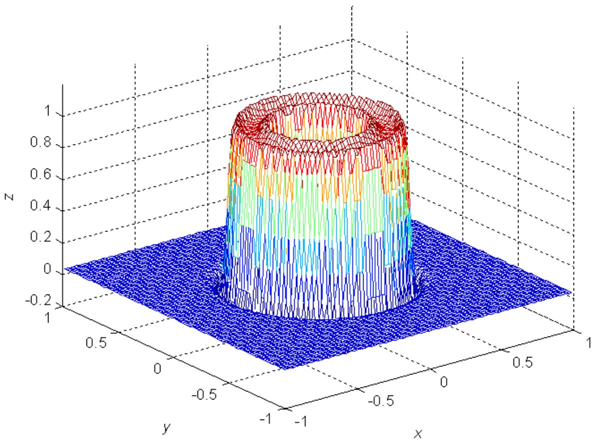

Fig. 8. Two-dimensional bandpass filter

4. CONCLUSION

With the method of compressed cosines two-dimensional filters with characteristics close to the ideal ones are obtained. It has to be noted that achieving optimal characteristic close to the ideal one requires a high resolution in the transition band of the filter. Otherwise, a Gibbs phenomenon is observed. The calculation of the spatial figure can be considerably alleviated, since in a large part of the passband and the stopband it has constant values $1-\varepsilon$ and ε, respectively (Fig. 4).

The proposed filters can find application to image processing with a high precision in such areas as astronomy, medicine, criminology, etc.

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