MODELING THE FLUID FLOW IN HUMAN INNER EAR

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Abstract

A mathematical model is developed related to the work of the cochlea based on the turbulent fluid movement equations. This is concluded from the basic equation of fluid dynamics in stress using so called effective viscosity. A characteristic equation is made on the basis of the resulting equations, as well as its components. The most common model of turbulence is used- $k-\varepsilon$.

Key words: *flows inbiological processes, cochlea, turbulentmotion, characteristicequation, modelofturbulence*

1. INTRODUCTION

When worked out the theoretical model is approached in a way that corresponds to the equations used in the program Fluent (Ansys). These equations differ from the usual used in Reynold's fluid mechanics equations [1]. The work out is made on the base of the main equation of the fluid mechanics in tensions [2]. Instead μ is replaced by μ_{eff} in processing equations (8). It is obtained a very convenient from of the system equations in a practical point of view. It is necessary to define some of the numbers [3] in the mentioned above equations of turbulence model and to replace them in the resulting system.

2. PHYSICAL DESCRIPTION OF THE FLOW IN COCHLEA

It is assumed that due to the impact of pounding the gavel on the oval window, arises an transient flow in cochlea tube which wraps the basilar membrane along its entire length, passes through the helicotrema to the bottom side, reaching the round window and back. Knowing that the cochlea fluid has the same properties as the water, it is easy to calculate the Reynolds number:

$$\mathrm{Re} = \frac{ud}{\upsilon} \tag{1}$$

In the pressure Δp created on the surface of the oval window the liquid is driven at a speed $u \approx 0,77 m/s$. The diameter of the cochlea is taken approximately as $d = 0,003m$; the viscosity of the liquid when the body temperature is $t = [36,5 \div 37]^o C$ then $\upsilon = 0,77.10^{-6} m^2/s$. In these parameters Re- number is obtained as $\mathrm{Re} \approx 3078$. This gives grounds to assume turbulent nature of the flow. Since this value is close to the critical Reynolds's number $(\mathrm{Re}_{critical} = 2320)$ in the resolve will require consideration of viscous stresses. Reason for this is the presence of boundary layers on the walls of the cochlea and the basilar membrane in a very small range of $2,5 \div 3mm$.

The flow is isothermal at a constant temperature or at slightly changing in the range from $t = 36,5 \div 40^o C$. In this situation, the density remains constant $(\rho = const)$. The flow is a liquid which in particular at a constant temperature is an incompressible fluid.

When recording of differential equations of continuity and motion is adopted:

$$\upsilon_t = \frac{\mu_t}{\rho} \tag{2}$$

$$\upsilon = \frac{\mu}{\rho} \tag{3}$$

i.e. they will appear on the coefficients of the kinematic viscosity υ and hence turbulent viscosity υ_t.

3. MATHEMATICAL MODEL OF THE FLOW

The mathematical model of the flow is based on the equation of continuity and the dynamics of fluids in tensions. In vector form they can be presented as [2]:

$$div\vec{V} = 0 \tag{4}$$

$$\rho\frac{d\vec{V}}{dt}+\rho\left(\vec{V}\vec{\nabla}\right)\vec{V}= \\ =\rho\vec{F}+\frac{\partial\vec{p}_x}{\partial x}+\frac{\partial\vec{p}_y}{\partial y}+\frac{\partial\vec{p}_z}{\partial z} \quad (5)$$

where $\vec{V}=u\vec{i}+v\vec{j}+w\vec{k}$ is 3 dimensional flow velocity; u,v,w are components of the velocity by x,y,z; $\vec{F}=X\vec{i}+Y\vec{j}+Z\vec{k}$ is the intensity of the field of mass force sand its relevant components along the axes; $\vec{p}_x,\vec{p}_y,\vec{p}_z$ are internal tensionsin the three axes; $\vec{\nabla}=\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k}$ is Hamptom operator.

According to summaried Newton's law is considered the relation between the tensor of the internal tension P and deformation speed S.

$$P=aS+b \quad (6)$$

At the conclusion of the equations in the case of turbulent flow is assumed:

$$a=2\mu_{eff} \quad (7)$$

$$\mu_{eff}=\mu+\mu_t \quad (8)$$

Thus determined tensor unit a difference with the usual Navier-Stokes' equations.

By using (7) and (8) for tangential and normal strains of turbulent flow of incompressible fluid is obtained:

$$\begin{aligned}
\tau_{yx}&=\tau_{xy}=(\mu+\mu_t)\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\\
\tau_{zx}&=\tau_{xz}=(\mu+\mu_t)\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\\
\tau_{yz}&=\tau_{zy}=(\mu+\mu_t)\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\\
\delta_x&=-p+2(\mu+\mu_t)\frac{\partial u}{\partial x}\\
\delta_y&=-p+2(\mu+\mu_t)\frac{\partial v}{\partial y}\\
\delta_z&=-p+2(\mu+\mu_t)\frac{\partial w}{\partial z}
\end{aligned} \quad (9)$$

After substituting (9) into (5) developed in scalar type by the respective axes is obtained as follows:

$$\begin{aligned}
&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=\\
&=-\frac{1}{\rho}\frac{\partial p}{\partial x}+2\frac{\partial}{\partial x}\left[(\upsilon+\upsilon_t)\frac{\partial u}{\partial x}\right]+\\
&+\frac{\partial}{\partial y}\left[(\upsilon+\upsilon_t)\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\\
&+\frac{\partial}{\partial z}\left[(\upsilon+\upsilon_t)\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]
\end{aligned} \quad (10)$$

$$\begin{aligned}
&\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=\\
&=-\frac{1}{\rho}\frac{\partial p}{\partial x}+2\frac{\partial}{\partial y}\left[(\upsilon+\upsilon_t)\frac{\partial v}{\partial y}\right]+\\
&+\frac{\partial}{\partial x}\left[(\upsilon+\upsilon_t)\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\\
&+\frac{\partial}{\partial z}\left[(\upsilon+\upsilon_t)\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]
\end{aligned} \quad (11)$$

$$\begin{aligned}
&\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=\\
&=-\frac{1}{\rho}\frac{\partial p}{\partial z}+2\frac{\partial}{\partial z}\left[(\upsilon+\upsilon_t)\frac{\partial w}{\partial z}\right]+\\
&+\frac{\partial}{\partial x}\left[(\upsilon+\upsilon_t)\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]+\\
&+\frac{\partial}{\partial y}\left[(\upsilon+\upsilon_t)\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]
\end{aligned} \quad (12)$$

Accordingly continuity equation:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \quad (13)$$

4. CHARACTERISTIC EQUATION

The problem is reduced to solving a 3 dimensional unsteady flow of a incompressible turbulent fluid. These equations ((10) - (13)) can be reduced to the characteristic equation:

$$\frac{\partial\Phi}{\partial t}+div(V\Phi)=div(\Gamma grad\Phi)+S \quad (14)$$

where: Φ is the relevant dependent variable; V accepted values u,v,w; Γ is diffusion coefficient; S is source article on relevant Φ. Values of Φ,V,Γ,S are given in Table 1.

The solution of (14) leads to the determination of the values of the three components u, v, w of the velocity and the pressure p in 3-dimensional unsteady flow over incompressible fluid. For this solving, it is necessary to determine the coefficient of kinematic turbulent viscosity υ_t. Fluent software have brought several models of turbulence, which allow the determination of: a modification of $k-\varepsilon, k-\Omega$ and models based on Reynolds tensions. In solving the problem is using a $k-\varepsilon$ model where:

$$\upsilon_t = c\mu \frac{k^2}{\varepsilon} \qquad (15)$$

where K is turbulent kinetic energy $\left(k = \frac{1}{2}\left(\overline{u}'^2 + \overline{v}'^2 + \overline{w}'^2\right)\right)$; ε is the speed of dissipation. Equations about k and ε have the expression:

$$\begin{aligned}&\frac{\partial}{\partial t}(\rho k) + \frac{\partial}{\partial x}(\rho k u) = \\ &= \frac{\partial}{\partial y}\left[\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial y}\right] + \\ &+ G_k + G_b - \rho\varepsilon - Y_M + S_K\end{aligned} \qquad (16)$$

$$\begin{aligned}&\frac{\partial}{\partial t}(\rho\varepsilon) + \frac{\partial}{\partial x}(\rho\varepsilon u) = \\ &= \frac{\partial}{\partial y}\left[\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial \varepsilon}{\partial y}\right] + \\ &+ C_{1\varepsilon}\frac{\varepsilon}{k}(G_k + C_{3\varepsilon}G_b) - C_{2\varepsilon}\rho\frac{\varepsilon^2}{k} + S_\varepsilon\end{aligned} \qquad (17)$$

The constants of the equations above are given in Table 2 [4], [5].

Because the flow is isothermal, and the fluid is incompressible $G_B = 0$ and $Y_M = 0$. The article G_K definds the generation of energy due to the interaction of Reynolds tensions and the gradient of the average speed is defined as:

$$G_K = -\overline{u'v'}\frac{\partial v}{\partial x} \qquad (18)$$

According to the Boussinesq approximation can determine:

$$G_K = \upsilon_t S^2 \qquad (19)$$

where S is the tensor of deformation speeds; the members S_K and S_ε are viscous dissipation.

Table 1. Coefficients the in characteristic equation

Φ	Γ	S
1	0	0
u	$\upsilon + \upsilon_t$	$\frac{\partial}{\partial x}\left(\Gamma\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\Gamma\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(\Gamma\frac{\partial w}{\partial x}\right) - \frac{\partial p}{\partial x}$
v	$\upsilon + \upsilon_t$	$\frac{\partial}{\partial x}\left(\Gamma\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(\Gamma\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(\Gamma\frac{\partial w}{\partial y}\right) - \frac{\partial p}{\partial y}$
w	$\upsilon + \upsilon_t$	$\frac{\partial}{\partial x}\left(\Gamma\frac{\partial u}{\partial z}\right) + \frac{\partial}{\partial y}\left(\Gamma\frac{\partial v}{\partial z}\right) + \frac{\partial}{\partial z}\left(\Gamma\frac{\partial w}{\partial z}\right) - \frac{\partial p}{\partial z}$

Table 2. Constants of the equations about k and ε

$C_{1\varepsilon}$	$C_{2\varepsilon}$	$C_{3\varepsilon}$	C_μ	σ_k	σ_ε
1,44	1,92	0,8	0,09	1.0	1.3

5. CONCLUSION

It is worked out equations related to the unsteady turbulent incompressible fluid flow in human inner ear (cochlea). These equations are very close to the Navier-Stokes equations. They are very convenient for numerical solution of the problem as they allow selection of the most suitable model of turbulence. This paper presents the characteristic equation with its components and used later $k-\varepsilon$ model of turbulence. The approach is a novelty in resolving similar biomedical problems.

References

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