ON SOME QUESTIONS CONCERNING TO THE MULTIFRACTAL SPECTRA CALCULATION

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Abstract

The methods of multifractal analyses are now widely used because they allow us to describe digital images with complex structure. For an image one may a set of fractal dimensions of its subsets – multifractal spectrum. There are two kinds of spectra – multifractal and Renyi spectra. Theoretically they are connected by the Legendre transform, but when working with experimental data this relation may not be true due to the dependence of results on many factors. Nevertheless the obtained multifractal characteristics may be successfully applied to image classification. We discuss some problems connected with spectra calculation and results of experiments.

1. INTRODUCTION

The number of high-resolution digital images having complex structure grows steadily in various areas of scientific exploration. These data are of great importance in biology, medicine, geology. Currently there is the tendency to apply fractal and multifractal methods for analysis and classification of such images, because these methods are more appropriate to describe complex textures.

Fractal methods are based on the assumption that a measure of an element of the image partition (cell or box) is approximately equal to the cell size to a power. Fractal sets may be characterized by one power, which is called also by scaling or singularity power. In particular capacity dimension is determined by this power. For multifractal sets, which are unions of different fractals, one can obtain a set of dimensions — multifractal spectrum. This characteristic may be used as a classification criterion for image analysis.

It should be noted that the calculation of multifractal characteristics is based on using so called statistical (information) approach. The natural measure distributed on the image is expressed in terms of pixel intensities. It is calculated for each cell, and then the measure is normed, so we obtain a probability distribution.

There are two spectra describing a multifractal set: the spectrum of Rényi dimensions and multifractal spectrum. Theoretically, there is a connection between them that is called the Legendre transform. But such is not necessarily the case for experimental data. The discussion of possible reasons and results of calculation for some classes of images are given in [4, 6].

In our practice we calculated both spectra for complex texture images from various classes of biomedical preparations. This work summarizes the obtainned results and discusses possible reasons of discrepancy between typical multifractal spectrum and the spectra constructed by experimental data.

2. MULTIFRACTAL SPECTRA, STATISTICAL SUM AND OTHERS

Textures are often may be considered as fractal (or multifractal) sets. Fractal sets have a self-similarity property that may be characterized by a scaling exponent. A union of several fractals, which of them has its own fractal dimension, is called multifractal. It is usually assumed that an image is partitioned by cells with size l , the number of cells is $N(l)$ and the measure of i-th cell

$$p_i(l) \sim l^{\alpha_i}. \quad (1)$$

The numbers α_i are called singularity (scaling) exponents. Then we combine cells with close exponent values in subsets and calculate fractal dimensions $f(\alpha_i)$. Hence we have a correspondence $(\alpha, f(\alpha))$. Fractal dimensions form so called multifractal spectrum.

In many cases the graphic of multifractal spectrum has a canonical parabolic form [3,5,6]. The situation may be illustrated for two-scale Cantor set or the attractor of baker transformation, when one can define a measure distribution by analytical way. But for complex textures the obtained graphic may differ from the canonical one considerably.

There is another kind of spectrum – Renyi dimensions. Consider a partition of an image, a measure $p_i(l)$, and the generalized statistical sum

$$\varphi(q,l) = \sum_{i=1}^{N(l)} p_i^q(l)\ , \tag{2}$$

where q is a real parameter.

We assume that there is such a function $\tau(q)$ that

$$\varphi(q,l) \sim l^{\tau(q)}. \tag{3}$$

Then $\tau(q) = \lim_{l\to 0} \frac{\ln \varphi(q,l)}{\ln l}$, and Renyi spectrum is defined as

$$D_q = \frac{1}{q-1}\tau(q). \tag{4}$$

For q = 0, 1, 2 we obtain capacity, information and correlation dimensions respectively.

One may obtain a connection between these spectra. We assume that the number of cells whose singularity exponents are in interval of the length α is also distributed by a power law, i.e. there is a distribution function $dn = l^{-f(\alpha)}d\alpha$. Then we substitute the cell measure in statistical sum (taking into account (1)), express the sum by using the distribution function as an integral, and for any q find a condition when the statistical sum is maximal. These actions result in obtaining the relation

$$f(\alpha) = \alpha q - \tau(q), \alpha = \tau'(q), \tag{6}$$

which is well known as the Legendre transform from variables $(q, \tau(q))$ to $\big((\alpha, f(\alpha)\big)$.

When $f(\alpha)$ and D_q are smooth function of α and q, $f(\alpha)$ may be obtained from this relation. But in practice we usually know D_q only in several points with variable precision, so the application of the Legendre transform is incorrect.

In this connection in [3] a method of the calculation of multifractal spectrum without using Legendre transform was proposed. Given an initial distribution $\{p_i(l)\}$, construct the sequence of measures $\mu(q,l) = \{\mu_i(q,l)\}$ obtained by the direct multifractal transform:

$$\mu_i(q,l) = \frac{p_i^q(l)}{\sum_{i=1}^{N} p_i^q(l)}. \tag{5}$$

Then we calculate singularity spectra and the Hausdorff dimensions of the supports of measures $\mu_i(q,l)$ as the functions of the parameter q of the statistical sum. Excluding q we obtain $f(\alpha)$.

3. METHODS OF CALCULATION

One may point out two methods for calculation of the spectrum.

3.1. Density function

This approach was proposed in [7] and used for image segmentation in [1,2].

We consider a special function (density function) to calculate the singularity power for every pixel. Then we combine all the pixels with close values of density function, which results in partition of the image on the subsets – so called level sets. For each level set we calculate its fractal dimension..

Let μ be a measure defined through pixel intensities. For $x \in R^2$ we denote $B(x,r)$ a square of length r with center x. Let $\mu(B(x,r)) = kr^{d(x)}(x)$, where $d(x)$ is the local density function of x and k some constant. Then

$$d(x) = \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

The density function measures the non-uniformity of the intensity distribution in the square $B(x,r)$. The set of all points x with local density α is a level set

$$E_\alpha = \{x \in R^2 : d(x) = \alpha\}$$

In practice, not to increase the number of level sets, one really consider sets

$$E(\alpha,\varepsilon) = \{x \in R^2, d(x) \in [\alpha, \alpha+\varepsilon)\}.$$

In [2] we constructed spectra for images of healthy liver and liver with a decease. Graphics did not have canonical form and depended from the measure choice considerably.

3.2. Using statistical sum

The second method uses statistical sum (2) and the sequence of measures (5) generated from the initial measure by the direct fractal transform [5]. The method was proposed in [3] and is based on the calculation of the Hausdorff dimension of a measure support M by the formula

$$\dim M = -\lim_{N\to\infty} \frac{\sum_{i=1}^{N} p_i \ln p_i}{\ln N} \qquad (7)$$

The direct multifractal transform results in a transformation of the initial measure by using statistical sum, and hence it depends on *q* as well. For any measure from the generated sequence one may calculate the singularity power averaged over the measure and the fractal dimension of the support of the measure corresponding to this singularity power. Hence we obtain the averaged singularity spectrum $\alpha(q)$, and the fractal dimension of the support of the measure $f(q)$ as functions of the parameter q. Eliminating *q* one can obtain the relation between singularity values and fractal dimensions of corresponding subset.

For each measure $\mu(q,l)$ one can calculate the Hausdorff dimension of its support by formula (7). As q changes, we have a set $f(q)$ of the Hausdorff dimensions of $\mu(q,l)$ supports, where

$$f(q) = \lim_{l\to 0} \frac{\sum_{i=1}^{N} \mu_i(q,l) \ln \mu_i(q,l)}{\ln l} = \lim_{l\to 0} \frac{f(q,l)}{\ln l}. \qquad (8)$$

We also calculate averaging exponents over the measure $\mu(q,l)$, i.e.

$$\sum_{i=1}^{N} \alpha_i \mu_i(q,l) = \frac{\sum_{i=1}^{N} ln\, p_i(l) \mu_i(q,l)}{ln\, l} = \frac{\alpha(q,l)}{ln\, l},$$

and then the limit $\alpha(q)$ of these averagings when $l \to 0$. Hence, we obtain

$$\alpha(q) = \lim_{l\to 0} \frac{\alpha(\mathrm{q},l)}{\ln l}. \qquad (9)$$

Such a method allows us to obtain the set of dimensions (multifractal spectrum) $f(q)$ and the set of averaging exponents $\alpha(q)$ as functions of the parameter *q*.

In practice, to obtain the above values by (8) and (9), we should do the following. For every *q* we take several values of variable l, calculate sets of points $(\ln l, f(q,l))$ and $(\ln l, \alpha(q,l))$ respectively. Then, by using the least square method, we determine the approximate values of $f(q)$ and $\alpha(q)$. Thus, we have the set of the Hausdorff dimensions of the supports of the measures that are obtained from the initial measure by the direct multifractal transform.

It is interesting to note that substituting $\mu_i(q,l) = \frac{p_i^q(l)}{\sum_{i=1}^{N} p_i^q(l)}$ in (8) we obtain

$$f(q) = \lim_{l\to 0} \frac{\sum_{i=1}^{N} \mu_i(q,l)\, ln\, \mu_i(q,l)}{ln\, l} =$$

$$\lim_{l\to 0} \frac{\sum_{i=1}^{N} \mu_i(q,l)\, ln \frac{p_i^q(l)}{\sum_j p_j^q(l)}}{ln\, l} =$$

$$q \lim_{l\to 0} \frac{\sum_{i=1}^{N} \mu_i(q,l)\, ln\, p_i(l)}{ln\, l} - \lim_{l\to 0} \frac{\sum_{i=1}^{N} \mu_i(q,l)\, ln\, \varphi(q,l)}{ln\, l} =$$

$$q\alpha(q) - \lim_{l\to 0} \frac{ln\, \varphi(q)}{ln\, l} = q\alpha(q) - \tau(q).$$

Besides that,

$$\frac{d\tau(q)}{dq} = \lim_{l\to 0} \frac{1}{ln\, l} \frac{\sum_i p_i^q(l)\, ln\, p_i(l)}{\sum_j p_j^q(l)} =$$

$$\lim_{l\to 0} \frac{\sum_i \frac{p_i^q(l)}{\sum_j p_j^q(l)} ln\, p_i(l)}{ln\, l} =$$

$$\lim_{l\to 0} \frac{\sum_i \mu_i(q,l)\, ln\, p_i(l)}{ln\, l} = \alpha(q).$$

Experiments were performed for Brodatz textures and various classes of biomedical preparation images. Practically in all cases the obtained graphics did not correspond the expected canonical form. We illustrate the situation for the image of pharmacological solution of Ag. (Fig.1)

The measure of a cell is the ratio of the sum of pixel intensities to the sum of intensities for the whole image. The calculations were performed for H component of HSV palette

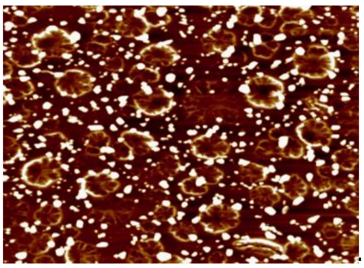

Fig. 1. Pharmacological solution of Ag

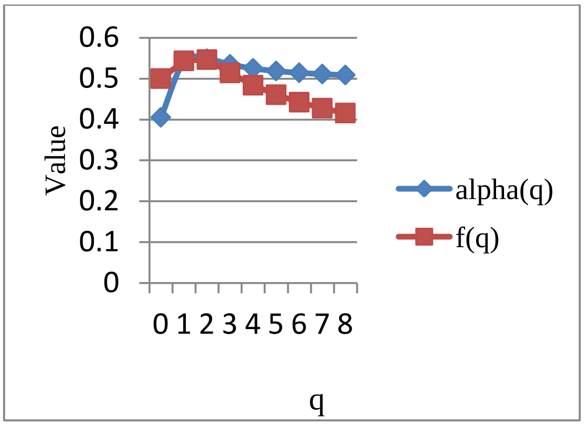

Fig. 2. Singularity and multifractal spectra

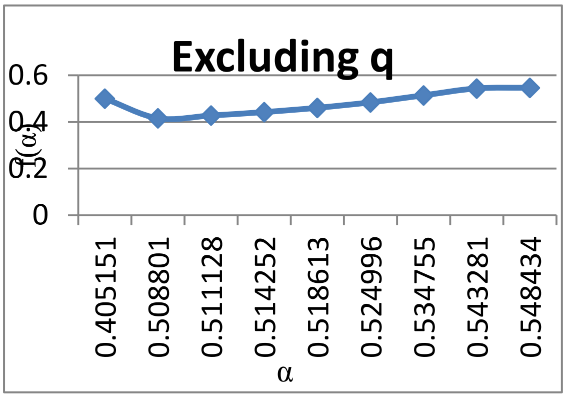

Fig. 3. Graphic $f(\alpha)$ obtained by excluding q

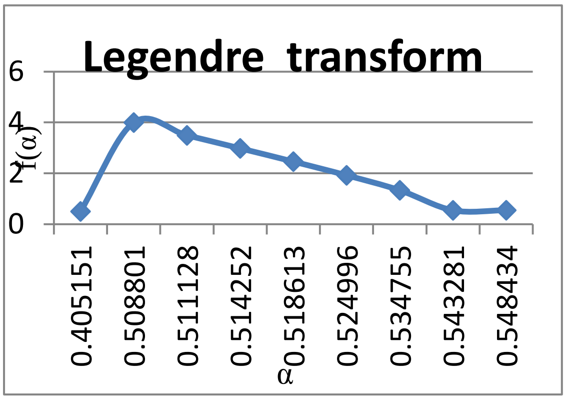

Fig. 4. Graphic $f(\alpha)$ obtained by Legendre transform

CONCLUSION

There are a lot of reasons that influence on the results. First and foremost the subsets of a multifractal are arranged by rather complicated manner, so that the problem of their separation may be solved only approximately. The cell measure may be calculated as the ratio the sum of pixel intensities of the cell to the common sum of intensities of the whole image, or as the ratio of the number of black pixels to the common number of black pixels (or the same for white ones), which means the implicit binarization of a grayscale image. For color images their reducing to grayscale ones may lead to a loss of the image structure, so we should consider the palette components separately. It is unknown a priori if the averaged characteristics may give a good approximation to the image measure. The question how to choose the range for q is also may be solved experimentally. It is easy to understand that there are no common rules to select the parameters for calculation, because they are defined by an image structure to a great extent.

Our experiments show that when using both the described methods we may not obtain canonical graphics due to dependence of results on many factors.

This circumstance should not prevent us from applying these techniques for calculation multifractal characteristics that are classification criteria in digital image analysis.

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