SOME APPROACHES TO FINDING ANALYTICAL FORM OF LIMIT CYCLES

Kostadin Sheiretsky

University of National and World Economy, Faculty of Applied Informatics and Statistics, Sofia, Bulgaria
E-mail: sheyretski@unwe.bg

Svetlin Antonov

Technical university of Sofia, Faculty of telecommunications, Sofia
E-mail: svantonov@yahoo.com , ORCID: 0000-0002-1698-8506

Abstract

A variant of the harmonic balance method is presented for finding a solution to limit cycle systems. It is shown how the solution can be constructed using the presence of a small parameter in the equation and in the absence of such a parameter. One way to prove loop stability is considered

1. INTRODUCTION

The change of a system can be traced [1] in phase space – a coordinate system formed by the variables and their velocities, and the qualitative behavior of the solution of the corresponding differential equation describing the system is called a phase portrait.

A closed phase portrait trajectory is called a limit cycle if there exists a tubular neighborhood of the trajectory that contains no other closed trajectories. There are three types of limit cycles: stable – the trajectories wind around the limit cycle on both sides of it, unsustainable – its trajectories spiral away from the limit cycle on both sides of it, and semi-stable – on one side the trajectories wind around the limit cycle, and on the other they move away from him.

2. A METHOD FOR FINDING AN ANALYTICAL FORM OF THE LIMIT CYCLE IN THE PRESENCE OF A SMALL PARAMETER

Let us look for a stationary solution of an equation of the type:

$$\ddot{x} + \omega^2 x + \varepsilon f(x, \dot{x}) = 0. \quad (1)$$

Let us look for the solution of the differential equation (1) in the form [2]:

$$x = x_1 + \beta + x_2 + x_3 + \cdots \quad (2)$$

$$x_i = a_i \cos i\,\psi + b_i \sin i\psi, \psi = k_0 t + \delta_0, \quad (3)$$

where a_i, b_i , k_0, δ_0 and β are constants for $i = 1..n$. In order to obtain an unambiguous decision, we will accept $b_1 = 0$. We impose the condition

$$\sum_{i=1}^{n}(\ddot{x}_i + (ik_0)^2 x_i) =$$
$$= \sum_{i=1}^{n}[(ik_0)^2 - \omega^2]\, x_i - \omega^2\beta - \quad (4)$$
$$-\varepsilon f\left(\sum_{i=1}^{n} x_i + \beta, \sum_{i=1}^{n} \dot{x}_i\right) = 0.$$

Let

$$S_1 = \sum_{i=1}^{n} x_i + \beta, S_2 = \sum_{i=1}^{n} \dot{x}_i.$$

In general, the sought quantities, decomposed in order by the powers of the small parameter, can be found using the system:

$$\left(k_0{}^2 - \omega^2\right)a_1 = \frac{\varepsilon}{\pi}\int_0^{2\pi} f(S_1, S_2)\cos\psi d\psi, \quad (5)$$

$$\int_0^{2\pi} f(S_1, S_2)\sin\psi d\psi = 0\ , \quad (6)$$

$$\omega^2\beta = -\varepsilon\int_0^{2\pi} f(S_1, S_2) d\psi, \quad (7)$$

$$a_j = \frac{\varepsilon}{\pi[(k_0\, j)^2 - \omega^2]}\int_0^{2\pi} f(S_1, S_2)\cos j\psi\, d\psi \quad (8)$$

$$b_j = \frac{\varepsilon}{\pi[(k_0\, j)^2 - \omega^2]}\int_0^{2\pi} f(S_1, S_2)\sin j\psi\, d\psi \quad (9)$$

$$j = 2,3, \dots n;$$

Let's look at the van der Pol equation as an example.

$$\ddot{x}+x-\varepsilon(1-x^2)\dot{x}=0, \qquad (10)$$

where ε is a small parameter. We will look for the solution in the form:

$$x=x_1+\beta+x_2+x_3+\cdots, \qquad (11)$$

as it must satisfy the condition:

$$\begin{gathered}\ddot{x}_1+{k_0}^2x_1+\\+\ddot{x}_2+4{k_0}^2x_2+\\+\ddot{x}_3+9{k_0}^2x_3=\\=\left({k_0}^2-1\right)x_1+\\+\left(4{k_0}^2-1\right)x_2+\\+\left(9{k_0}^2-1\right)x_3-\beta+\\+\varepsilon\left(1-{x_s}^2\right)\frac{d}{dt}x_s=0,\\x_s=x_1+\beta+x_2+x_3. \qquad (12)\end{gathered}$$

Due to the presence of the first derivative on the right-hand side of the first equality, we will choose x_i as follows:

$$\begin{gathered}x_i=a_i\cos i\,\psi+b_i\sin i\psi,\\\psi=k_0t+\delta_0,\\b_1=0\,. \qquad (13)\\i=1..n.\end{gathered}$$

By substituting in the above equality, after the necessary calculations, it is established that a stationary process is possible at the following values of the quantities:

$$k_0=1, a_1=2, a_2=0, b_2=0, a_3=0,$$
$$b_3=\varepsilon\frac{{a_1}^3}{32}, \beta=0. \qquad (14)$$

3. CONSTRUCTING A SOLUTION BY THE HARMONIC BALANCE METHOD WITHOUT APPARENT DEPENDENCE ON A SMALL PARAMETER

Let's find an approximate solution to the equation.

$$\ddot{x}-\dot{x}(1-3x^2-2\dot{x}^2)+x=0\,. \qquad (15)$$

We will look for the solution of equation (15) as a sum of two terms:

$$x=x_1+x_3. \qquad (16)$$

Since preliminary estimates give $a<1$, this amplitude can be assumed to be a small parameter, a circumstance that we only consider in order to break the order to $O(a^4)$. We write both terms in the form:

$$\begin{gathered}x_1=a\cos\psi,\\x_3=a^3\mu_1\sin 3\psi+a^3\mu_2\cos 3\psi,\\\psi=k_0t+\gamma. \qquad (17)\end{gathered}$$

Where $\mu_1,\ \mu_2,\ ,\gamma$ and k_0 are constants to be determined by the harmonic balance method.

We translate the main equation (15) into the form:

$$\begin{gathered}\ddot{x}_1+{k_0}^2x_1+\ddot{x}_3+9{k_0}^2x_3=\\=\left({k_0}^2-1\right)x_1+\left(9{k_0}^2-1\right)x_3+\\+(\dot{x}_1+\dot{x}_3)[(1-3(x_1+x_3)^2-\\2(\dot{x}_1+\dot{x}_3)^2)]=0. \qquad (18)\end{gathered}$$

Equation (18) leads to solving only algebraic equations. Substituting into the equation accordingly, we get:

$$\left({k_0}^2-1\right)a\cos\psi=0, \qquad (19)$$

$$ak_0\left(1-\frac{3a^2}{4}-\frac{3a^2{k_0}^2}{2}\right)\sin\psi=0, \qquad (20)$$

$$\left[\left(9{k_0}^2-1\right)\mu_1a^3-3\mu_2a^3-ak_0\left(-\frac{3a^2}{4}+\frac{a^2{k_0}^2}{2}\right)\right]\sin 3\psi=0 \qquad (21)$$

$$\left[\left(9{k_0}^2-1\right)\mu_2a^3+3\mu_1a^3\right]\cos 3\psi=0. \qquad (22)$$

After calculation it is obtained:

$${k_0}^2=1, a=\frac{2}{3}, \mu_1=-\frac{2}{73}, \mu_2=\frac{3}{292}. \qquad (23)$$

Due to the fact that the harmonic balance method generally does not use the small parameter concept, the solution can be improved by adding new terms and breaking the order to our chosen harmonic functions, which can easily be done by computer.

4. LIMIT CYCLE STABILITY ANALYSIS

An important role in the study of limit cycles is played by the concept of a positive invariant set - this is a set such that, if we choose an arbitrary point of it as a starting point, then during the further evolution of the system, the trajectory will remain in this set [1].

Let us study systems admitting limit cycles, which generally have the mathematical expression:

$$\ddot{x}+\omega^2x=F(x,\dot{x}). \qquad (24)$$

To investigate the system we will put:

$$x = a(t)\cos\psi, \psi = \omega t + \delta(t). \quad (25)$$

We also accept:

$$\dot{a}\cos\psi - a\dot{\delta}\sin\psi = 0,$$
$$\dot{x} = -\omega\, a\sin\psi. \quad (26)$$

By substituting into equation (25), the system is reached:

$$\dot{a} = -\frac{1}{\omega}F(a\cos\psi, -\omega\, a\sin\psi)\sin\psi,$$
$$\dot{\delta} = -\frac{1}{a\omega}F(a\cos\psi, -\omega\, a\sin\psi)\cos\psi. (27)$$

If a limit cycle exists, it will be a closed curve with equation:

$$F(a\cos\psi, -\omega\, a\sin\psi) = 0. \quad (28)$$

This fact can become clearer if we consider the following statement derived on the basis of a statement presented in [3].

Let $F(a\cos\psi, -\omega\, a\sin\psi)$ be a continuous function of its variables. If there exist two real positive numbers a_1 and a_2 such that $a_1 < a_2$ and the conditions are met:

$$-F(a\cos\psi, -\omega\, a\sin\psi) > 0, \text{ if } a < a_1$$
$$-F(a\cos\psi, -\omega\, a\sin\psi) < 0, \text{ if } a > a_2$$

for any value of ψ, there exists a positive invariant set $M = \{x | a_1 < a < a_2\}$.

The condition 1 shows that: $\dot{a} > 0$ for $a < a_1$, and the condition 2 shows respectively, that: $\dot{a} < 0$ for $a > a_2$ and since F is continuous with respect to a, it will exist $a^*(\psi)$, such that $F(a^*(\psi)\cos\psi, -\omega\, a^*(\psi)\sin\psi) = 0$, for any fixed ψ . The fact is obvious that for amplitude values $a_1 < a < a_2$, the phase trajectory "winds" on the closed curve described by the equation

$$F(a^*(\psi)\cos\psi, -\omega\, a^*(\psi)\sin\psi) = 0$$

when $\psi \in \left[0, \frac{2\pi}{\omega}\right]$.

As a consequence, the fact can be established that at $a_1 > a_2$ the limit cycle will be unstable.

Finding a_1 and a_2 can be done by considering that:

$$a_1 = min\{a | F(a, \psi) = 0\};$$
$$a_2 = max\{a | F(a, \psi) = 0\}. \quad (29)$$

Let's take just an example:

$$\ddot{x} + x = \dot{x}(1 - \lambda_1 x^2 - \lambda_2 \dot{x}^2),$$
$$\lambda_1 > \lambda_2 > 1. \quad (30)$$

Taking into account the conditions for finding the limits of the positive invariant set, we arrive at the equations:

$$a_1 = \frac{1}{\sqrt{max\left\{\frac{\lambda_1+\lambda_2}{2} + \frac{\lambda_1-\lambda_2}{2}\cos 2\psi\right\}}},$$
$$a_2 = \frac{1}{\sqrt{min\left\{\frac{\lambda_1+\lambda_2}{2} + \frac{\lambda_1-\lambda_2}{2}\cos 2\psi\right\}}}. \quad (31)$$

We may take specific values for $\lambda_1 = 3$ and $\lambda_2 = 2$. Then it is easy to define the positive invariant set:

$$M = \left\{x \middle| \frac{1}{\sqrt{3}} < a < \frac{1}{\sqrt{2}}\right\}. \quad (32)$$

A result that differs slightly from the estimates of [1]:

$$M = \left\{x \middle| \frac{1}{2} < a < \frac{1}{\sqrt{2}}\right\}. \quad (33)$$

Due to the simple structure of the equation, we can immediately write down the equation of the limit cycle:

$$a^*(\psi) = \frac{\sqrt{\frac{2}{\lambda_1+\lambda_2}}}{\sqrt{1 + \frac{\lambda_1-\lambda_2}{\lambda_1+\lambda_2}\cos 2\psi}}. \quad (34)$$

Very rarely can one arrive at an analytical expression for the limit cycle equation. In most cases, it is necessary to use asymptotic methods to find an approximate solution [4, 5, 6, 7]. Such use is in help in the area of Telecommunications, Fluid dynamics and Fire protection analysis.

5. ACKNOWLEDGMENTS

The authors would like to thank the Research and Development Sector at the Technical University of Sofia for the financial support.

References

[1] Arrowsmith, D. Place, C. Ordinary Differential Equations. A Qualitative Approach with Applications. Chapman and Hall, London, New York 1982.

[2] O. Blacker, Analysis of Nonlinear Systems [Russian translation], Mir, Moscow, 1969.

[3] Nemytski V., V. Stepanov. Qualitative Theory of Differential Equations, Princeton University Press 1960.

[4] Nayfeh, A., Perturbation methods, John Wiley&Sons, Inc., ISBN:9780471399179, Ney York, 2000.

[5] Bogoliubov N., Mitropolsky A., Asymptotic Methods in the Theory of Non-Linear Oscillations. New York, Gordon and Breach, 1961.

[6] Andronov A. A., A. A. Vit, S. E. Haikin. Theory of oscillations, Fizmatgiz, M.,1981.

[7] Sheiretsky, K.G., Antonov, S., Criteria for the existence of a sustainable limit cycle. application of the criteria in the first approximation for the van der pol equation, 56th International Scientific Conference on Information, Communication and Energy Systems and Technologies, ICEST 2021 - Proceedings, ISBN: 978-1-6654-2888-0, DOI: 10.1109/ICEST52640.2021.9483481, 2021, pp. 127-129.