ON SOME CASES OF USING THE SMALL PARAMETER METHOD FOR FINDING PERIODIC SOLUTIONS

Kostadin Sheiretsky

University of National and World Economy, Faculty of Applied Informatics and Statistics, Sofia, Bulgaria
E-mail: sheyretski@unwe.bg

Svetlin Antonov

Technical university of Sofia, Faculty of telecommunications, Sofia
E-mail: svantonov@yahoo.com , ORCID: 0000-0002-1698-8506

Abstract

A modified small parameter method is presented in which the zero-th approximation solution is used to construct the first approximation solution. Simple but practically important differential equations, respectively autonomously and non-autonomously, are considered, showing the application of the method. In the non-autonomous equation, the non-resonant and resonant case are discussed.

1. INTRODUCTION

In the case of nonlinear differential equations, the presence of a small parameter allows to search for a periodic solution in asymptotic order by the powers of this parameter [1]. Usually, such a parameter is embedded in the structure of the equation itself, for example, it can be multiplied before the perturbing term [2, 3]. We will consider a case in which we will look for the decomposition of the solution in order of the degrees of the amplitude [1]. Such an approach can easily be set by a small deviation initial condition for the autonomous differential equation. In the non-autonomous equation, for the non-resonant case, the amplitude will depend on the periodic effect on the oscillator, and if we choose it to have a small amplitude, we will again ensure correctness of the problem. At resonance, the amplitude increases significantly and the asymptotic expansion must be done for a small parameter involved in the structure of the equation [4]. In constructing the asymptotic series itself, we choose the first approximation terms to be a power function of zero approximation. Such use is in help in the area of Telecommunications, Fluid dynamics and Fire protection analysis.

2. AUTONOMOUS CASE

When considering the problem of the movement of the pendulum, for small deviations from the equilibrium position, the formula [3,4] is obtained:

$$\ddot{x} + \omega^2 x - \frac{\omega^2}{6} x^3 = 0. \qquad (1)$$

In the equation, ω^2 is a constant quantity. Let's set the initial conditions:

$$x(0) = l, \dot{x}(0) = 0. \qquad (2)$$

Since we will be considering small deviations in the equilibrium position, the amplitude l can be used as a small parameter. We do the laying:

$$\Omega t = \tau, \frac{d^2}{dt^2} = \Omega^2 \frac{d^2}{d\tau^2}, \Omega = const \qquad (3)$$

and the derivative with respect to variable τ is denoted by ex:

$$\Omega^2 x'' + \omega^2 x - \frac{\omega^2}{6} x^3 = 0. \qquad (4)$$

We will look for the solution in the species:

$$x = l\xi_0 + l^3 \xi_1 + O(l^5). \qquad (5)$$

We also decompose Ω by the powers of the small parameter:

$$\Omega = \omega + l^2 \omega_1 + O(l^4). \qquad (6)$$

We substitute lines (5) and (6) in the differential equation (4) and arrive at the expressions with the first degree in the small parameter and the third degree in the small parameter, respectively:

$$l: \xi_0'' + \xi_0 = 0, \qquad (7)$$

$$l^3: \xi_1'' + \xi_1 = -2\frac{\omega_1}{\omega} \xi_0'' + \frac{1}{6} \xi_0{}^3. \qquad (8)$$

We impose the conditions:

$$\xi_0(0) = 1, \xi_0(0) = 0;$$
$$\xi_1(0) = 0, \xi_1{}'(0) = 0. \quad (9)$$

The solution to the first equation can immediately be determined:

$$\xi_0 = \cos\tau. \quad (10)$$

We will look for the solution of equation (8) in the form:

$$\xi_1 = \lambda\xi_0{}^3 - \lambda\xi_0. \quad (11)$$

We find the second derivative of this function and use that:

$$\xi_0{}'^2 = 1 - \xi_0{}^2, \quad (12)$$

the final result looks like this:

$$\xi_1'' = 6\lambda\xi_0 - 6\lambda\xi_0{}^3 - 3\lambda\xi_0{}^3 + \lambda\xi_0. \quad (13)$$

Substituting into the equation gives:

$$\lambda = -\frac{1}{48}, \omega_1 = -\frac{1}{16}\omega. \quad (14)$$

The final solution of the problem in the approximation adopted by us is written in the form:

$$\Omega = \omega\left(1 - \frac{l^2}{16}\right), \quad (15)$$
$$x = l\xi_0 + l^3\left(-\frac{1}{48}\xi_0{}^3 + \frac{1}{48}\xi_0\right),$$
$$\xi_0 = \cos\Omega\mathrm{t}. \quad (16)$$

3. A NON-AUTONOMOUS CASE

Let us consider the non-autonomous differential equation [1,4]:

$$\ddot{x} + \omega^2 x - \frac{\omega^2}{6}x^3 = F\cos pt. \quad (17)$$

The quantities ω^2 and F are constants.

We will consider only the particular solution when the oscillations are created solely by the perturbing force $F\cos pt$. As in the previous case, we will use the deflection amplitude for a small parameter, this requires the constraint F to be small compared to unity and furthermore the oscillation frequency to be far from the resonance frequencies.

We introduce a new variable:

$$pt = \tau, \quad (18)$$

and substitute in the differential equation (17), noting the derivatives with respect to the new variable with ex:

$$p^2 x'' + \omega^2 x - \frac{\omega^2}{6}x^3 = F\cos\tau. \quad (19)$$

We will look for the solution of (19) in the form:

$$x = q\xi_0 + q^3\xi_1 + O(q^5). \quad (20)$$

For the function ξ_1 we will take the expression:

$$\xi_1 = \lambda_1\xi_0{}^3 - \lambda_2\xi_0. \quad (21)$$

We substitute expressions (20) and (21) in the differential equation (19) and determine the equations that contain the first power of the amplitude and the third power of the amplitude, respectively. For the first equation we get:

$$qp^2\xi_0'' + (\omega^2 q - F)\xi_0 = 0. \quad (22)$$

Having accepted that:

$$\xi_0 = \cos\tau. \quad (23)$$

From here it immediately follows that:

$$q = \frac{F}{\omega^2 - p^2}. \quad (24)$$

The values of the unknown parameters we are looking for are also obtained from the equation containing the third degrees of amplitude:

$$6\lambda_1 = \frac{\omega^2}{-9p^2 + \omega^2}, \lambda_2 = -\frac{p^2\omega^2}{(-9p^2 + \omega^2)(\omega^2 - p^2)}. \quad (25)$$

When we consider the resonant case, we can no longer use the amplitude as a small parameter. Again we consider the case when a small parameter ε appears in the differential equation:

$$p^2 x'' + \omega^2 x - \varepsilon\omega^2 x^3 = \varepsilon F_0\cos\tau. \quad (26)$$

In equation (26), the derivative is taken with respect to the variable $\tau = pt$. We assume that the frequencies ω and p are close to each other and the relationship between them is carried out by the equation:

$$\omega = p + \varepsilon p_1. \quad (27)$$

We substitute ω in equation (26) and look for x in the form:

$$x = x_0 + \varepsilon x_1, x_1 = \lambda x_0{}^3. \quad (28)$$

For the zero approximation we get:

$$x_0'' + x_0 = 0. \quad (29)$$

We immediately determine that:

$$x_0 = B\cos\tau. \quad (30)$$

By substituting into the formula for a first approximation, we find the value of λ, as well as the equation for the relationship between the quantities B, p_1 and F_0:

$$\lambda = -\frac{1}{8}, \quad (31)$$

$$-\frac{3}{8}p^2B^3 + pp_1B - \frac{F}{2} = 0. \quad (32)$$

4. CONCLUSION

The presented methodology can easily be applied for further approximations. Since approximations to the first or second power of the small parameter are usually used for analytical purposes, the calculations made have a fully justified practical significance [3, 5]. A peculiarity of the presented methodology is that it analytically describes only the stationary process, but not the transient phenomena. This is a characteristic feature of the small parameter method, but nevertheless, for the study of oscillating systems, this method is classical [1]. Numerous applications of this method are the basis of the analytical approach to nonlinear differential equations used in various physical, biological, technical and other models [6, 7]. The authors apply this knowledge especially in Fluid dynamics, Optical and Radio communications, Tele-Medical communications etc. Our approach to this tool is related to the search for a solution as a power function of the zero approximation. The obtained results show that such an approach is justified and can be constructed, the way in which this can be done is also described.

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