

# Analysis of Piecewise Uniform Lattice Vector Quantization for Memoryless Laplacian Sources

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**Abstract** – In this paper we will present a procedure design of an optimal piecewise uniform vector quantizer of memoryless Laplacian source. We will derive the expressions for the granular distortion and the optimum number of the output points in each subregion. We will also analyze results presented in paper [3]. We will show that the assumptions used in [3] are not valid, causing an error when calculating the number of output points.

**Keywords** – Vector quantizer, Laplacian source

## I. INTRODUCTION

Quantizers play an important role in theory and practice of modern day signal processing. It was shown in the literature that vector or multidimensional quantization can yield smaller average mean squared error per dimension than scalar quantization for the case of fine quantization. Designing an optimal vector quantizer is equivalent to finding a partition of the vector space and assigning a representative point to each partition such that a predefined distortion measure between input and output is minimized. Unfortunately, the optimal partitions in higher dimensional spaces are unknown for even the simplest source distributions and the most common distortion measures. Extensive results have been developed on finding the optimal output points distribution in multidimensional space for a specific probability distribution. In a number of papers vector quantization of memoryless Laplacian source was analyzed.

The analysis of vector quantizer for arbitrary distribution of the source signal was given in paper [1]. The authors derived the expression for the optimum granular distortion and optimum number of output points. However, they didn't prove the optimality of the proposed solutions. Also, they didn't define the partition of the multidimensional space into subregions. In paper [2], they have derived the expressions for the optimum number of output points, however the proposed partitioning of the multidimensional space for memoryless Laplacian source doesn't take into consideration the geometry of the multidimensional source. In paper [3], vector quantizers of Laplacian and Gaussian sources were analyzed. The proposed solution for the quantization of memoryless Laplacian source, unlike in [2], takes into consideration the geometry of the source, however, the proposed vector quantizer design procedure is too complicated and impractical.

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In this paper we will give a systematic analysis of piecewise uniform vector quantizer of Laplacian memoryless source. We will give a general and simple way for the piecewise uniform vector quantizer. We will derive the optimum number of output points and the optimality of the proposed solutions will be proved.

## II. ASYMPTOTIC ANALYSIS

Consider a multidimensional piecewise uniform quantizer, where the input space is partitioned into subregions. Each subregion is divided using cubic cells with different size. Let the  $n$ -dimensional vector of Laplacian random variables be the input to the  $n$ -dimensional quantizer. For  $n$ -dimensional vector  $\mathbf{x} = [x_1 \ x_2 \ \dots \ x_n]^T$  consisting of i.i.d. Laplacian variables  $x_i$  with zero mean and unit variance, the joint pdf of  $\mathbf{x}$  is:

$$f(\mathbf{x}) = \prod_{i=1}^n f(x_i) = 2^{-\frac{n}{2}} \exp\left(-\sqrt{2} \sum_{i=1}^n |x_i|\right) \quad (1)$$

The contour of constant pdf is given by:

$$\sum_{i=1}^n |x_i| = -\frac{1}{2\sqrt{2}} \ln(2^n f_c^2) \equiv r_0, \quad r_0 > 0$$

This is an expression for the  $n$ -dimensional hyperpyramid with radius  $r_0$ , where we define the radius as:

$$r = \sum_{i=1}^n |x_i| \quad (2)$$

Since each  $|x_i|$  has an exponential distribution with mean  $1/\sqrt{2}$  and variance  $1/2$  the random variable  $r$  has a gamma pdf given as:

$$f_n(r) = K_n r^{n-1} e^{-\sqrt{2}r}, \quad r \geq 0 \quad (3)$$

where

$$K_n = \frac{2^{\frac{n}{2}}}{\Gamma(n)}$$

The probability that the vector  $\mathbf{x} = [x_1 \ x_2 \ \dots \ x_n]^T$  is in the  $k$ th cell denoted by  $P_k$  is equal to:

$$P_k = \int_{r_k}^{r_{k+1}} f_n(r) dr \quad (4)$$

where  $r_k$  and  $r_{k+1}$  denote the radius of the  $k$ th and  $(k+1)$ th region  $R_k$  and  $R_{k+1}$ , respectively.

Granular distortion for piecewise vector quantization of the signal generated by the Laplacian source can be written as follows:

$$D_g = \frac{1}{12} \sum_{k=0}^{N_q-1} \Delta_k^2 P_k = \frac{1}{12} \sum_{k=0}^{N_q-1} \left( \frac{V_k}{N_k} \right)^{\frac{2}{n}} P_k \quad (5)$$

where the volume  $V_k$  of the  $k$ th subregion, is given by the following equation:

$$V_k = \frac{2^n r_{k+1}^n}{\Gamma(n+1)} - \frac{2^n r_k^n}{\Gamma(n)} \quad (6)$$

where the  $r_k$  and  $r_{k+1}$  denote radial parameter. By using the Langrangian multipliers from equation (5) we can obtain the optimum number of cells in one subregion  $N_k$ :

$$J = D_g + \lambda \sum_{k=0}^{N_q-1} N_k \quad (7)$$

After differentiating  $J$  with respect to  $N_k$ ,  $k=0, N_q-1$ , and equalizing with zero, with some mathematical manipulation, we obtain the number of points in the  $k$ th region as:

$$N_k = N \frac{V_k^{\frac{2}{n+2}} P_k^{\frac{n}{n+2}}}{\sum_{i=0}^{N_q-1} V_i^{\frac{2}{n+2}} P_i^{\frac{n}{n+2}}} \quad (8)$$

with the respect to the condition that the total number of output points is equal to:

$$N = \sum_{k=0}^{N_q-1} N_k \quad (9)$$

By substituting  $N_k$  from equation (8) in equation (5) we have:

$$D_g = \frac{1}{12N^{2/n}} \left( \sum_{k=0}^{N_q-1} P_k^{\frac{n}{n+2}} V_k^{\frac{2}{n+2}} \right)^{\frac{n+2}{n}} \quad (10)$$

The total distortion can be calculated using the following expression:

$$D = \frac{1}{12N^{2/n}} \left( \sum_{k=0}^{N_q-1} P_k^{\frac{n}{n+2}} V_k^{\frac{2}{n+2}} \right)^{\frac{n+2}{n}} + \frac{1}{n} \int_{r_{N_q}}^{\infty} \left[ (r - r_{N_q})^2 + \frac{(n-1)\Delta_{N_q}^2}{12} \right] f_n(r) dr \quad (11)$$

where  $r_{N_q}$  denotes the radius of the input space. In [3] a codebook dilution was proposed where a codebook is constructed by a series of successive subregions filled with different sets of cubic cells of length  $c_k$ . It was shown that for

the minimum distortion the following equation must be satisfied:

$$c_B = \exp \left[ \frac{\sqrt{2}(r_B - r_A)}{n+2} \right] \cdot c_A \quad (12)$$

where  $c_A$ ,  $c_B$ ,  $r_A$  and  $r_B$  represent the lengths of cells and radii of two successive subregions. They proposed a codebook dilution where the lengths of cells in two successive subregions satisfy the following constraint:

$$c_{k+1} = s \cdot c_k \quad (13)$$

where  $s$  represents scale factor ratio. If we denote the length of cells in the first subregion as  $c_0$  from (13) it follows that

$c_k = c_0 s^k$ . If we put in equation (12)  $r_A=0$ ,  $c_A = c_0$ ,  $r_B = r_k$  and  $c_B = c_k$  we than have:

$$\exp \left( \frac{\sqrt{2}r_k}{n+2} \right) = s^k \quad (14)$$

Solving (14) for  $r_k$ , we obtain the radial parameter:

$$r_k = \frac{(n+2)\ln s}{\sqrt{2}} \cdot k, \quad k=0, N_q-1 \quad (15)$$

Next, we will show that the method proposed in [3] for calculating the number of the output points in one subregion is not correct. We will start from the equation:

$$N = \frac{c_0^{-n}}{n!} \left[ \sqrt{2}(n+2)\ln s \right]^n \sum_{k=0}^{N_q-1} s^{-kn} \left( (k+1)^n - k^n \right) \quad (16)$$

From previous equation we can express  $c_0$  as:

$$c_0 = \frac{\sqrt{2}(n+2)\ln s}{(n!N)^{1/n}} \left[ \sum_{k=0}^{N_q-1} s^{-kn} \left( (k+1)^n - k^n \right) \right]^{1/n} \quad (17)$$

The difference between equations (16) and (17) and equations used in [3] is that in [3] sumations are infinite. The number of the output points in each subregion was calculated in [3] as:

$$N_k = \frac{V_k}{(s^k c_0)^n} \quad (18)$$

In [3] they use the following algorithm in order to evaluate the performance of the proposed uniform piecewise vector quantizer. First, for a given bit rate, the best scale factor ratio  $s$  is chosen experimentally, and the initial value of  $c_0$  is calculated using (17) with infinite summation. After that, they begin constructing integer subcodebooks from the center outward by calculating the number of output points in each subcodebook, until the required number of output points is reached for the whole codebook. The number of output points  $N_k$  obtained this way, is than substituted in the expression similar to the equation (11).

The length of the first cell  $c_0$  can be obtained in other way by substituting equation (16) in equation (9):

$$N = \sum_{k=0}^{N_q-1} N_k = \frac{1}{c_0^n} \sum_{k=0}^{N_q-1} V_k s^{nk} \quad (19)$$

From equation (19) we have:

$$c_0 = \frac{1}{N^{1/n}} \left( \sum_{k=0}^{N_q-1} \frac{V_k}{s^{nk}} \right)^{1/n} \quad (20)$$

Using equation (1) and equation (14) we can express  $s^{-k}$  as:

$$s^{-k} = 2^{\frac{n}{2(n+2)}} f^{\frac{1}{n+2}}(r_k) \quad (21)$$

If we substitute  $s^{-k}$  from equation (21) in equation (20) we can write:

$$c_0 = \frac{2^{\frac{n}{2(n+2)}}}{N^{1/n}} \left( \sum_{k=0}^{N_q-1} V_k f^{\frac{n}{n+2}}(r_k) \right)^{1/n} \quad (22)$$

If we substitute (22) and (21) in (19) we obtain the elegant way of calculating the number of output points in each subregion without the need for any simulation or vagueness:

$$N_k = N \frac{V_k f^{\frac{n}{n+2}}(r_k)}{\sum_{i=0}^{N_q-1} V_i f^{\frac{n}{n+2}}(r_i)} \quad (23)$$

Now, let us assume that the pdf  $f(r)$  is constant over the valume of the  $k$ th subregion  $V_k$ . In that case we can substitute  $P_k$  with  $P_k = V_k f(r_k)$ . Equation (5) can be written as follows:

$$D_g \approx \frac{1}{12} \sum_{k=0}^{N_q-1} \left( \frac{V_k}{N_k} \right)^{2/n} V_k f(r_k) \quad (24)$$

Using the Lagrangian multipliers we can determine the optimum number of the output points and distortion as we did earlier. In this way we can show that in this case the optimum number of the output points is given with the equation (23), and the distortion with the equation:

$$D_g \approx \frac{1}{12N^{2/n}} \left( \sum_{k=0}^{N_q-1} f^{\frac{n}{n+2}}(r_k) V_k \right)^{\frac{n+2}{n}} \quad (25)$$

Results given by the equations (23) and (25) are based on the assumption that  $P_k = V_k f(r_k)$ . The problem is that the former assumption is not valid for the bit rates per dimension and the dimensions used in [3] and in this paper. Therefore, the results given by the equations (23) and (25) are not correct. In [3] they use  $N_k$  given with the equation (23) and for the distortion they use equation (5). Results obtained this way are close to the optimum, but because of the use of  $N_k$  given by (23), still not correct.

The number of output points given with equations (8) and (23) is shown in Fig. 1. It was used uniform piecewise vector quantizer with  $N_q = 6$  subregions, the dimension  $n = 16$  and the bit rates per dimension  $R = 1$  and  $R = 2$ .

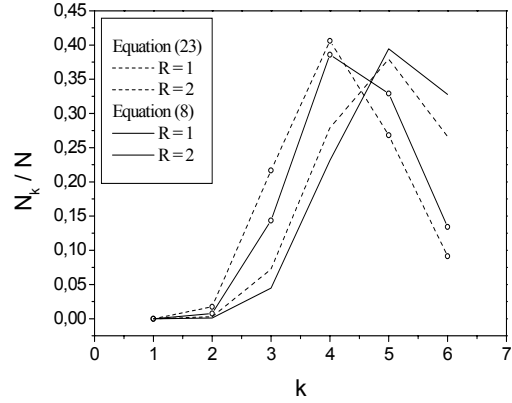


Fig. 1. The number of output points.

As we can see from the Fig. 1. the number of output points calculated using (23) differ significantly from the optimum number of the output points calculated using (8). The difference between these two curves occurs because  $f(r_k)V_k$  differs significantly from  $P_k$  for optimum input load.

In Table 1. the values of the distortion calculated using (10), (25) and the results obtained in [3] are given.

TABLE I

R	$r_{opt}$	$N_{q,opt}$	D (10)	D (25)	D [3]
1	15.3652	6	3.534E-1	2.833E0	2.937E-1
2	20.0918	11	8.940E-2	3.949E-1	9.840E-2
3	23.9199	20	2.200E-2	5.605E-2	2.382E-2
4	26.8136	33	5.459E-3	1.011E-2	5.875E-3
5	29.6395	61	1.359E-3	1.945E-3	1.459E-3

As we can see from Table I the assumption that  $P_k \approx f(r_k)V_k$  becomes valid at the higher bit rates per dimension. The difference between  $P_k$  and  $f(r_k)V_k$  is demonstrated in Table II for bit rates per dimension  $R = 1$  and  $R = 2$ . For  $R = 1$  the optimum number of suregions is eqaul  $N_q = 6$ , and for  $R = 2$  that number is  $N_q = 11$ .

Distortion as a function of the number of subregions  $N_q$ , for  $n = 16$ , is plotted in Fig. 2.

The total distortion plottet in Fig. 2. is caculated according to equation (10). The length of each region is  $r_{opt}/N_q$ , where  $r_{opt}$  represents the optimum input load, the length of cells in each subregion must be  $\Delta_k \leq r_{opt}/N_q$ . The length of cells increases with  $k$  so the most critical is the length of cells in the last subregion. When the cell length in the last subregion

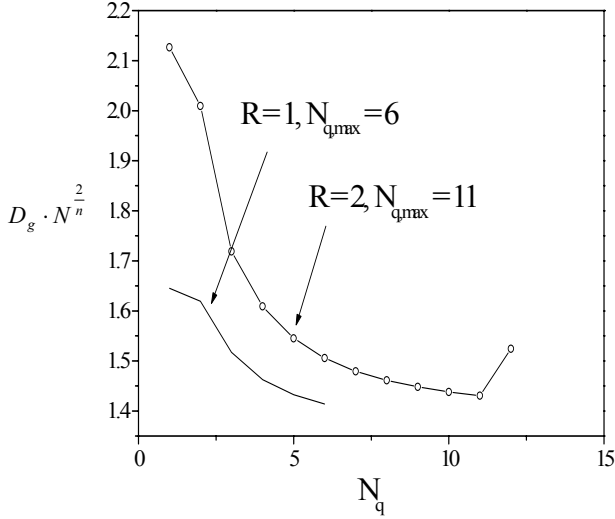


Fig. 2. Total distortion as a function of  $N_q$ .

TABLE II

$k$	$R = 1$		$R = 2$	
	$P_k$	$f(r_k)V_k$	$P_k$	$f(r_k)V_k$
1	1.4169E-6	4.1861E-5	1.6697E-8	1.8778E-7
2	3.3305E-3	7.3357E-2	9.9902E-5	9.2958E-4
3	8.2238E-2	1.28646E0	6.1041E-3	4.6054E-2
4	2.9389E-1	3.40297E0	5.5056E-2	3.4415E-1
5	3.4230E-1	3.17356E0	1.6771E-1	9.0668E-1
6	1.9311E-1	1.52704E0	2.5388E-1	1.2325E0
7			2.3688E-1	1.061E0
8			1.5532E-1	6.5423E-1
9			7.7931E-2	3.1287E-1
10			3.1713E-2	1.2252E-1
11			1.0907E-2	4.0839E-2

$\Delta_{N_q-1}$  becomes larger than  $r_{opt}/N_q$  than we must use input load  $r_{max}$  that satisfies the condition

$$\Delta_{N_q-1} = r_{max}/N_q \quad (26)$$

This causes the degradation of the vector quantizer and the distortion increases. For  $R = 1$  the solution of the equation (26) does not exist, and the maximum value of the subregions is equal to  $N_{q,max} = 6$ .

Distortion as a function of the input load and the dimension of the vector quantizer is plotted in Fig. 3.

As we can see the distortion decreases, and the optimum input load increases as the dimension of the vector quantizer increases.

### III. CONCLUSION

In this paper we have presented a procedure design of an optimal piecewise uniform vector quantizer of memoryless Laplacian source. We derived the expressions for the granular

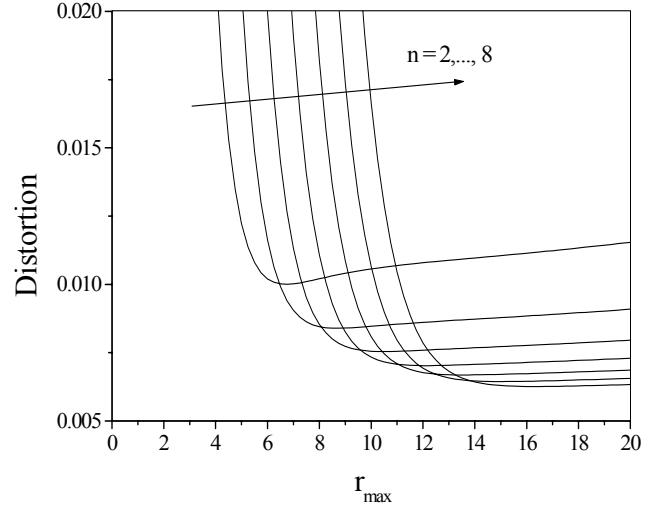


Fig. 3. Influence of the dimension of the vector quantizer on the distortion

distortion and the optimum number of the output points in each subregion. Unlike in papers [1] and [2], using Lagrangian multipliers we proved the optimality of the proposed solutions. We also analyzed results presented in [3]. Design procedure proposed in [3] is complicated and unpractical. The desing procedure of piecewise uniform quantizer we have presented is based on minimum distortion criterion. From this we then derive the optimum number of output points. Unlike in [3], where they use experiments to obtain the necessary parameters, we use closed forms to calculate all parameters of the quantizer. Also, we have shown that the expression for the optimum output points obtained in [3] is not correct, and that it corresponds to the optimum solution under the assumptions that  $P_k$  can be approximated with  $f(r_k)V_k$ . These assumption, used in [3] is not valid, causing an error when calculating the number of output points.

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