Stabilization of Interconnected Systems via Math-Analytical Approach: Lyapunov Second Method Design

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Abstract—**The crucial role of Lyapunov second method in designing decentralized stabilization control of composite interconnected systems with delay inter-connections via math-analytical approach is studied. By assumption, solely measurable outputs of the interconnected plants are available, hence output feedback control problem is solved. Interconnections are assumed to satisfy the so-called matching conditions, and a sufficient condition is derived that ensures uniform ultimate boundeness for both subsystems and the overall system via Lyapunov method. This result implies practical local stability of the underlying system dynamics.**

***Keywords*—Composite interconnected systems, electrical power systems, Lyapunov design, output feedback.**

I. INTRODUCTION

In the theory of composite interconnected systems, it has been demonstrated that traditional control theory approaches via the centralized control concept is almost of no practical applicability and effectiveness. Consequently, the decentralized control approach for such systems is a crucial design problem, which has been studied extensively for more than two decades. Many results have been given [1]-[3], all of them making use of Lyapunov second method one way or another. However, the problem of decentralized stabilization of composite interconnected systems with delays, in general, has not been paid the appropriate attention. Nonetheless, it is important to prove that resulted controls do guarantee asymptotic or at least practical stability [3], [5] at the design stage, and not to be investigated and ensured a-posteriori trough simulation experiments [10].

The presence of delay phenomenon, which is generally the case in many practical engineering systems such as electrical power systems, hydraulic pressure transmission gearing, and rail transportation systems etc. due to transport phenomena and computation time involved, has been known to be the main obstacle [1]-[6], [8], [11]. It is well known that in composite systems, sources of uncertainties and delays appear to be either within the individual subsystems or in the interconnection links among subsystems [4]-[6]. In the present paper, which extends the previous results in [11] uncertainties and delays are assumed to appear in the interconnections, and for this case the stabilization control problem is resolved by using the second method of Laypunov stability theory. A design solution to decentralized output feedback controller that guarantees the uniform ultimate boundedness of every subsystem and of the overall system. Hence the practical asymptotic stability in the sense of local Laypunov stability is also ensured.

The paper is organized as follows. The preliminary analysis and problem statement are presented in Section II. Then in Section III, there is give a lemma and some known analysis of the stabilization control studied along with the new theorem. The proof is omitted due to paper size limits, and therefore Laypunov function synthesized and its time derivative as well as some useful inequalities are given as hints. Conclusion and references follow thereafter.

II. PROBLEM STATEMENT AND PRELIMINARY ANALYSIS

A. System Representation Model

The following representation model of a class composite interconnected systems with interconnection delays, which is composed of N interconnected sub-systems S_i, is described as follows:

$$S_i: \quad \begin{aligned} \dot{\boldsymbol{x}}_{\boldsymbol{i}}(\boldsymbol{t}) &= Ax_i(t) + Bu_i(t) + \\ &+ \sum_{j=1, j\neq i}^{N} BH_{ij}(x_j(t), x_j(t-\tau), t), \\ y_i(t) &= Cx_i(t). \end{aligned} \qquad (1)$$

It should be noted that, in here, subsystem variables and quantities in Eqs. (1) are realistic representations, respectively, denoting: $x_i \in R^{n_i}$ is the state vector; $u_i \in R^{r_i}$ is the control (input) vector; $y_i \in R^{m_i}$ is the output vector. A, B, C are the constant matrices of appropriate dimensions

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of system S_i realization; and τ is the uncertainty delay in the interconnections.

B. Assumptions Characterizing System Representation

System (1) is presumed to satisfy the realistic assumptions presented below.

Assumption 1: For each subsystem, the pair (A, B) is completely controllable (standard in modeling for control problems).

Assumption 2: There exist non-negative scalar constants β_{ij}, γ_{ij}, such that

$$\left\| H_{ij}\left(x_j(t), x_j(t-\tau), t\right)\right\| \le \gamma_{ij}\left\|x_j(t)\right\| + \\ + \gamma_{ij}\left\|x_j(t-\tau)\right\| + \beta_{ij} \tag{2}$$

Assumption 3: For every subsystem (A, B), there exists a matrix $F \in R^{m\times l}$, such that

$$T(s) = FC(sI - A)^{-1}B \tag{3}$$

is *strictly feedback positive real* (SFPR).

From Kalman-Yakubovich Lemma [7], it follows that there exist matrices $P \in R^{n\times n}$, $Q \in R^{n\times n}$, $K \in R^{m\times n}$, $P = P^T > 0$, $Q = Q^T > 0$, satisfying

$$(A + BK)^T P + P(A + BK) = -Q \tag{4}$$

$$\mathrm{Re}\left[\lambda(A + BK)\right] < 0 \tag{5}$$

such that

$$FC = B^T P. \tag{6}$$

Definition 1 [8]: Consider a system described by

$$\dot{x}(t) = f(t, x(t)) \tag{7}$$

where, $t \in R$ is the time variable, $x(t) \in R^n$ is the state. The system is said to be *uniform ultimate boundedness* with respect to S, if for any given set S and any $r \in (0, \infty)$, there exists a $T(S, r) \in [0, \infty)$ such that for every solution

$$x(\cdot)\colon [t_0, \infty) \to R^n,\ x(t_0) = x_0,$$

and for all $t \ge t_0 + T(S, r)$,

$$\left\|x_0\right\| \le r \Rightarrow x(t) \in S$$

holds.

As far as the uniform ultimate boundedness of systems is concerned, the following general previous result is available.

Lemma 1 [9]: Consider a general delay function differential equation

$$\dot{x}(t) = f(t, x(t)) \tag{8}$$

and the initial condition

$$x(t) = \varphi(t), \qquad t \in [t_0 - \tau,\ t_0]. \tag{9}$$

Suppose functions $r_1(\cdot)$ and $r_2(\cdot)$ belong to class K_∞, and $r_3(\cdot)$ belong to class K. System (8)-(9) is to possess uniform ultimate boundedness with respect to a finite constant $T(S, r)$, if the system has the following properties:

there exists a function $V(\cdot)\,:\,[t_0 - \tau, \infty) \times R^n \to R^+$ of class C^1 satisfying

(i) for all $t \in [t_0 - \tau, \infty)$, $x \in R^n$,

$$r_1\left(\|x\|\right) \le V(x, t) \le r_2\left(\|x\|\right) \tag{10}$$

(ii) there exists a constant $q > 1$ satisfying

$$\|x(\xi)\| < q\|x(t)\|, \qquad t - \tau \le \xi \le t, \quad t \ge t_0 \tag{11}$$

such that

$$\dot{V}(t, x) \le -r_3\left(\|x\|\right) + \varepsilon \tag{12}$$

where $\varepsilon > 0$ is a positive number, and satisfies

$$\varepsilon < \lim_{z\to\infty}\mathrm{inf} r_3(z) = c. \tag{13}$$

III. Design of Stabilizing Control and Main Result

As pointed out in the introduction, we limite ourselves to the problem of uniform ultimate boundedness of this class of composite systems by designing output feedback stabilization control in order to demonstrate that Lyapunov second method is indispensable. This is the focus in the sequel, and further discussion shows that the other-vise non-solvable output feedback stabilization problem is resolved via synthesizing an appropriate Laypunov function.

With regard to Assumption 3, it is possible to select a scalar $\mu > 0$ such that there exists a positive definite matrix $P \in R^{n\times n}$ satisfying Laypunov type of equation

$$A^T P + PA - \mu PBB^T P = -Q \tag{14}$$

Also, it should be noted with regard to conditions (4) and (5), the solution of (13) is guaranteed by choosing the following feedback gain matrix:

$$K = -\frac{1}{2}\mu B^T P.$$

Then, on the grounds of the novel stability Theorem 1 presented further below, the proposed decentralised feedback control scheme can be constructed as follows. For each $i \in N$, let

$$u_i = p_i(y_i(t),t) = -\frac{1}{2}k_i F y_i(t) \tag{15}$$

where, F is gain matrix, k_i is positive scalar.

$$k_i = \mu + \sum_{j=1, j\neq i}^{N} \left[\delta^2 \beta_{ij}^2 + 2d^2\gamma_{ij}^2\right], \quad i = 1,2,\ldots ,N \tag{16}$$

where, δ, d are positive scalars, and d is determined such that

$$\frac{1}{d^2} < \frac{\lambda_{\min}(Q)}{2(N-1)}. \tag{17}$$

The main new result, derived via the second method of Laypunov stability theory, is given in terms of the following theorem.

Theorem 1: Under Assumption 1-3, for the composite interconnected system (1) with uncertainty delay interconnections, the decentralised output feedback controls $u_i(t),\ i = 1,2,\cdots,N$, Eqs. (15), guarantee the uniform ultimate boundednes for each subsystem S_i and the overall system.

As a kind of hints for the proof, note the following points. Note first that and using output feedback control (15) For every subsystem S_i, system S_i can be rewritten as follows:

$$S_i: \quad \begin{aligned} \dot{\boldsymbol{x}}_{\boldsymbol{i}}(\boldsymbol{t}) &= Ax_i(t) + Bp_i(y_i(t),t) \\ &+ \sum_{j=1, j\neq i}^{N} BH_{ij}(x_j(t), x_j(t-\tau), t) \\ y_i(t) &= Cx_i(t). \end{aligned} \tag{18}$$

The chosen Lyapunov functions are

$$V(x_i(t)) = x_i^T(t) P x_i(t) \tag{19}$$

the same for every subsystem, where P is the solution of the equation (14). Based on Rayleigh principle, it can be shown that

$$\lambda_{\min}(P)\|x_i(t)\|^2 \le V(x_i(t)) \le \lambda_{\max}(P)\|x_i(t)\|^2 \tag{20}$$

Therefore $V(x(t)) = \sum_{i=1}^{N} V(x_i(t))$, and let $x_i(t)$ be the solution of (18). From Eqs. (15) and (18), the derivative of $V(x_i(t))$ along any trajectory of the closed-loop system (18) satisfies:

$$\begin{aligned} \dot{V}(\boldsymbol{x}_{\boldsymbol{i}}(\boldsymbol{t})) &= x_i^T(t)[A^T P + PA]x_i(t) \\ &- k_i x_i^T(t) PBFCx_i(t) \\ &+ 2x_i^T(t) PB \sum_{j=1, j\neq i}^{N} H_{ij}(x_j(t), x_j(t-\tau), t). \end{aligned} \tag{21}$$

Next, on the grounds of Lemma 1, for any $q > 1$, there exists

$$\|x_i(\xi)\| \le q\|x_i(t)\|, \qquad t-\tau \le \xi \le t.$$

By using the well known inequality $2ab \le a^2 + b^2$ for any real numbers a, b, it can be shown that

$$\begin{aligned} \dot{V}(x(t)) \quad \le \quad & -\sum_{i=1}^{N} \lambda_{\min}(Q)\|x_i(t)\|^2 \\ & + \sum_{i=1}^{N} \left[(N-1)\left(\frac{1+q^2}{d^2}\|x_i(t)\|^2 + \frac{1}{\delta^2} \right) \right] \end{aligned}$$

For the subsystems, it should be observed that the respective Lyapunov functions satisfy

$$\begin{aligned} \dot{V}(x_i(t)) \quad \le \quad & -\lambda_{\min}(Q)\|x_i(t)\|^2 \\ & + (N-1)\left(\frac{1+q^2}{d^2}\|x_i(t)\|^2 + \frac{1}{\delta^2} \right) \end{aligned}$$

Thus, the last inequality can be rewriten as follows.

$$\dot{V}(x_i(t)) \quad \le \quad -\omega_i\|x_i(t)\|^2 + \varepsilon_i,$$

where ω_i, ε_i express corresponding terms. Should the control parameter d is chosen to satisfying (17), then there exists a sufficiently small scalar $q > 1$, such that $\omega_i > 0$. On the other hand, for any $\delta > 0$, it is implied that $\varepsilon \le \lim_{z\to\infty} \inf r_{3i}(z)$, where $r_{3i}(\|x_i(t)\|) = \omega_i\|x_i(t)\|^2$, Thus, by virtue of Lemma 1, it can be shown that each subsystem is uniformly ultimately bounded. Hence, by Eq. (18) and the definition of $V(x(t))$, the overall system is also uniformly ultimately bounded.

IV. Conclusion

This paper is focused on solving the stabilization control design problem for a class of composite interconnected systems with uncertainty delay interconnections by using Lyapunov second method, which is not solvable otherwise. Thus it demonstrates the universality of Laypunov stability theory in general, and its potential as a design tool in particular. Although the new stability proof guarantees uniform ultimate boundedness of both each subsystem and the overall system, precisely because of the latter it also guarantees practical system stability. It is important to note that the designed controllers do not contain uncertainty time delay as a parameter.

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