On Conditions for Attractor Existence at Nonlinear Discrete Systems

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Abstract - **The conditions for existence of attractors at a class of nonlinear discrete systems are demonstrated in this paper. Results are illustrated by examples and confirmed by simulations.**

Keywords - **Nonlinear discrete systems, Attractors.**

I. Introduction

Mathematical models of nonlinear discrete systems have many applications in different branches of science: physics, chemistry, biology, economics, etc. The conditions for the existence of attractors in these systems are necessary but no sufficient [1]. The performance of necessary and sufficient conditions is possibly only for the narrow classes of nonlinear systems [2-4]. The conditions exhibited in this paper are more general than some known results, but they are weaker than conditions, which concern the narrow classes of systems. The approximate method for easy determination of the existence or absence of attractors in nonlinear discrete systems is given, too.

II. Necessary conditions for the existence of attractors

In order to determine rigorous conditions for the existence of attractors, we introduce the nonlinear discrete system in the next form:

$$\sum_{i=0}^{n} l_i(x(k))x(k+n-i)=0,\, l_0(x(k))=1 \tag{1}$$

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where:

$$l_i(x)=\varphi_i(x(k))=\varphi_i(x) \tag{2}$$

are nonlinear functions of state $x(k)$.

We will perform corresponding necessary conditions for the existence of attractors in nonlinear discrete systems. Instead of Eq. (1), we will consider a linear difference equation:

$$\sum_{i=0}^{n} l_i x(k+n-i)=0,\, l_0=1 \tag{3}$$

where l_i are constant parameters.

The stability area of Eq. (3), S_n, in parametric space $(l_1,l_2,...,l_n)$ is easy determined by some of known criteria, for example Schur-Cohn criterion.

Let us define a curve l in parametric space with a set of equations:

$$l \Leftrightarrow \begin{matrix} l_1=\varphi_1(x) \\ l_2=\varphi_2(x) \\ \vdots \\ l_n=\varphi_n(x). \end{matrix} \tag{4}$$

This is an oriented curve in the direction of increase $|x|$, starting from $x=0$. Let us denote with λ the set of points of the curve (4), and with $\vec{L}$ the unit tangent vector of curve l in the point $(l_1,l_2,...,l_n)$.

The necessary conditions for the existence of attractors and sufficient conditions for the absence of attractors can be expressed by the following theorem.

Theorem 1: The necessary conditions for the existence of attractors of system (1) are:

a)
$$\begin{gathered} S_n \cap \lambda \neq \varnothing \\ \vec{L}\,\mathrm{grad}\partial\,\mathrm{S_n} < 0 \end{gathered} \tag{5}$$

where $\vec{L}=\left(\frac{\partial\varphi_1}{\partial x},\frac{\partial\varphi_2}{\partial x},...,\frac{\partial\varphi_n}{\partial x}\right)$, while $\varnothing$ is the empty set.

b) The sufficient condition for the absence of attractors of system (1) is:

$$S_n \cap \lambda = \varnothing. \tag{6}$$

Since the area S_n is determined by the set of non-equalities, mostly linear ones, the limit surface ∂S_n is not differentiable in the lines of intersection of hyper planes. This interferes analytical determination of $\mathrm{grad}\partial S_n$ in any point. Therefore the intersection of the curve (4) and the corresponding limit plane is determined firstly and then a normal vector $\vec{N}$ on the plane in the point of intersection is determined. In this way the vector $\vec{N}$ is used instead of $\mathrm{grad}\partial S_n$ in relation (5).

The application of conditions (5) for the establishment of attractors in practical employ is sometimes complicated. The application of this theorem can be made easier in the following way. Let us define the area P_n (hyper parallelepiped) in the parametric space:

$$P_n \Leftrightarrow |l_i| \le \binom{n}{i}, i = 1, 2, ..., n \tag{7}$$

and the area O_n (polyhedron):

$$O_n \Leftrightarrow |l_1| + |l_2| + \ldots + |l_n| \le 1. \tag{8}$$

Then holds: **Theorem 2:**

$$\begin{aligned} &\text{a) } S_n \in P_n \\ &\text{b) } O_n \in S_n \ . \end{aligned} \tag{9}$$

The conditions (9) means that the area S_n is included in hyper parallelepiped (7) and that the area O_n is inside the area S_n. The necessary conditions for the existence of attractors mean that the curve (4) must intersects both P_n and O_n.

III. The Second Order Discrete Systems

A. Necessary Conditions

In the case of the second order systems, the Eq. (1) becomes:

$$\sum_{i=0}^{2} l_i(x(k))x(k+2-i) = 0, l_0(x(k)) = 1 \tag{10}$$

The corresponding linear equation is:

$$x(k+2) + l_1 x(k+1) + l_2 x(k) = 0 \tag{11}$$

i.e.

$$\begin{aligned} x_1(k+1) &= x_2(k) \\ x_2(k+1) &= -l_1 x_2(k) - l_2 x_1(k) \end{aligned} \tag{12}$$

Notice that initial conditions in the phase plane (x_{10}, x_{20}) will be mapped to initial conditions in the parameter plane (l_{10}, l_{20}), while the equilibrium point of system (10) (x_{1e}, x_{2e}) will be mapped to a equilibrium point in parameter plane (l_{1e}, l_{2e}).

The stability area S_2 is determined by relations:

$$S_2 \Leftrightarrow (l_2 > -l_1 - 1, l_2 > l_1 - 1, l_2 \le 1). \tag{13}$$

and shown in Fig. 1.

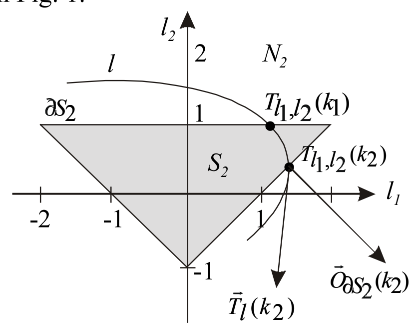

Fig. 1. Stability area S_2

Eq. (11) has stable solutions if parameters l_1, l_2 belong to the interior S_2 of the curve ∂S_2. A region of nonstability N_2 is exterior of the curve ∂S_2. System (11) will be stable if the perturbations of the coefficients of differential equation remain in the stability area. Let us define a curve l in parametric plane with a set of equations:

$$l \Leftrightarrow \begin{aligned} l_1 &= \varphi_1(x) \\ l_2 &= \varphi_2(x) \end{aligned} \tag{14}$$

Let $\vec{O}_{\partial S_2}(k)$ be the orthogonal vector of ∂S_2 in $(l_1(x(k)), l_2(x(k)))$, $\vec{T}_l(k)$ tangent vector of curve l in $(l_1(x(k)), l_2(x(k)))$ (Fig. 1). Let us also orient the curve l such that $l_{l_1(x_1), l_2(x_1)} \prec l_{l_1(x_2), l_2(x_2)}$ if $|x_1| < |x_2|$. We are also going to define function $\Psi(l_1(x_0), l_2(x_0))$ in the following way: $\Psi(l_1(x_0), l_2(x_0)) = \begin{cases} 1 & (l_1(x_0), l_2(x_0)) \in S_2 \\ -1 & (l_1(x_0), l_2(x_0)) \in N_2 \end{cases}$.

Theorem 3: The necessary conditions for the existence of stable attractors are:

1. The equilibrium point (l_{1e}, l_{2e}) must belong to the region of non-stability of the parametric plane, $(l_{1e}, l_{2e}) \in N_2$

2. l intersects the bounding curve ∂S_2, i.e., $l \cap \partial S_2 \neq \varnothing$, determining (at least one) points of intersection $T_{l_1, l_2}(k_i)$, $i = 1, 2, ..., n$.
3. There exist a set of points $T_{l_1, l_2}(k_i)$, $i = 1, 2, ..., n$ for which index:

$$K = \operatorname{sgn}[\Psi(l_1(x_0), l_2(x_0)) < \vec{O}_{\partial S_2}(k_i), \vec{T}_l(k_i) >] > 0$$

where $< \vec{O}_{\partial S_2}(k_i), \vec{T}_l(k_i) >$ stands for dot product of vectors $\vec{O}_{\partial S_2}(k_i)$ and $\vec{T}_l(k_i)$.

It is obvious that attractors can be formed only when the system is in points $T_{l_1, l_2}(k_i)$, $i = 1, 2, ..., n$

B. Simulation Example

In this subsection we are going to give simulation results to demonstrate the theoretical results obtained in preceding subsection.

Let us consider the following nonlinear discrete system:

$$\begin{aligned} & x(k+2) + l_1 x(k+1) + l_2 x(k) = 0, \\ & l_1 = l_1(x(k)) = x^2(k) - 1,\ l_2 = l_2(x(k)) = \text{const}. \end{aligned} \tag{15}$$

i.e.

$$\begin{aligned} & x_1(k+1) = x_2(k) \\ & x_2(k+1) = -l_1 x_2(k) - l_2 x_1(k) \end{aligned} \tag{16}$$

We are going to investigate existence of attractors of system (15) by changing value of coefficient l_2. The initial conditions in the phase plane; $x_1(0) = 0, x_2(0) = 0.9$, will be mapped to initial conditions in the parameter plane $l_1(0) = x_1^2(0) - 1 = -1$, $l_2(0) = const$. The searching in the parameter plane (l_1, l_2) will be done according to Fig. 2.

For $l_2 > 0.98$ and $l_2 < -1.3$ the system (15) is nonstable. For $l_2 = 0.98$ the chaos appears in the system (Fig. 3) and for $l_2 = 0.89$ the attractor is obtained.

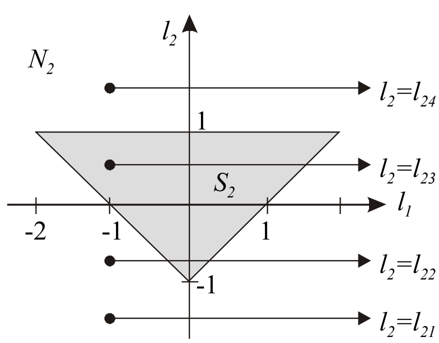

Fig. 2. The searching in the parameter plane (l_1, l_2)

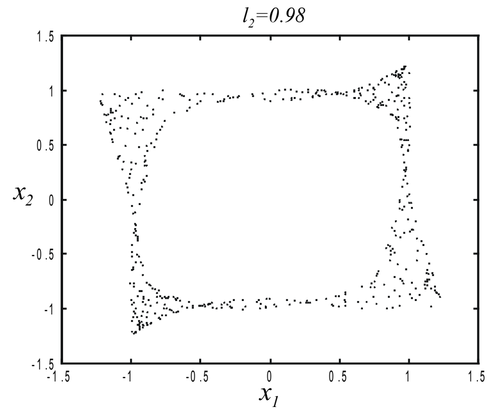

Fig. 3. Phase portrait of system (15) with $l_2 = 0.98$

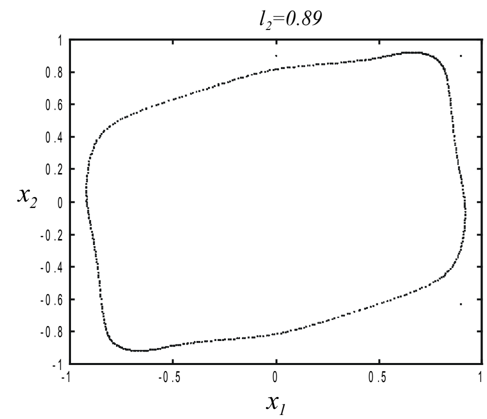

Fig. 4. Phase portrait of system (15) with $l_2 = 0.89$

IV. THE THIRD ORDER DISCRETE SYSTEMS

In the case of the third order systems, the Eq. (1) is:

$$\sum_{i=0}^{3} l_i(x(k)) x(k+3-i) = 0,\ l_0(x(k)) = 1. \tag{17}$$

The corresponding linear equation is:

$$x(k+3) + l_1 x(k+2) + l_2 x(k+1) + l_3 x(k) = 0\ . \tag{18}$$

The stability area S_3 is determined by relations:

$$\begin{aligned} & l_1 + l_2 + l_3 > -1 \\ & l_1 - l_2 + l_3 < 1 \\ & l_1 l_3 + 1 > l_2 + l_3^2\ . \end{aligned} \tag{19}$$

In the case of the third order system the area P_3 is parallelepiped determined by nonequalities:

$$P_3 \Leftrightarrow |l_1| \leq 3, |l_2| \leq 3, |l_3| \leq 1. \tag{20}$$

The O_3 is octahedron determined by nonequality:

$$O_3 \Leftrightarrow |l_1| + |l_2| + |l_3| \leq 1. \tag{21}$$

The curve l is given with:

$$\begin{aligned} l_1 &= \varphi_1(x) \\ l_2 &= \varphi_2(x) \\ l_3 &= \varphi_3(x) \, . \end{aligned} \tag{22}$$

The application of the exhibited method can be illustrated by an example:

$$\begin{aligned} & x(k+3) + l_1 x(k+2) + l_2 x(k+1) + l_3 x(k) = 0 \\ & l_1 = -x^2(k) + \frac{25}{16}, l_2 = 0.5, l_3 = \text{const.} \end{aligned} \quad . \tag{23}$$

The attractors of the system (23) for $l_3 = 0.6$ are shown in the Fig. 5. The attractor of the system (23) for $l_3 = -0.3$ is given in the Fig. 6. For $l_3 = -0.42$ the attractor is degenerated and has chaotic structure (Fig. 7). The motion is made in the third order phase space.

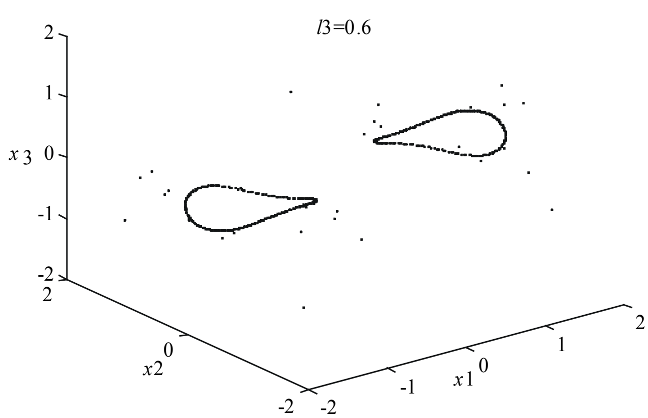

Fig. 5. Phase portrait of system (23) with $l_3 = 0.6$

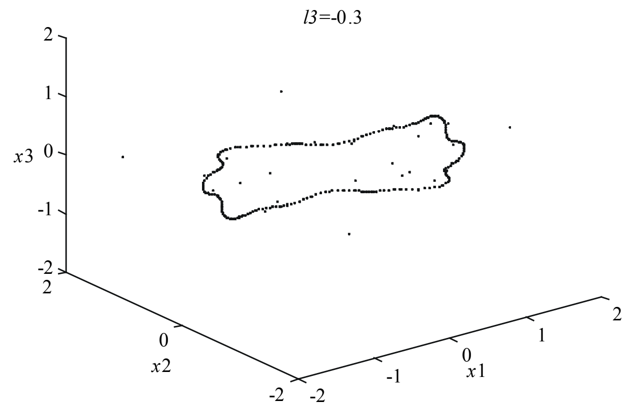

Fig. 6. Phase portrait of system (23) with $l_3 = -0.3$

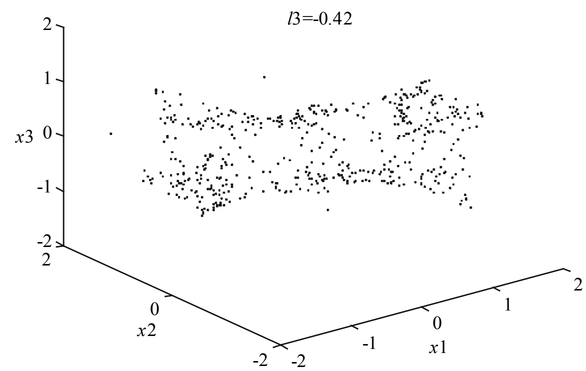

Fig. 7. Phase portrait of system (23) with $l_3 = -0.42$

V. Conclusion

The method presented in this paper enables determination of the necessary conditions for the existence of attractors in *n*-order nonlinear discrete systems in a clean algebraic manner. The verification of the given results is made by simulation. The existence of the attractors in the second and third order nonlinear discrete systems and appearance of the chaos in these systems are illustrated by concrete examples.

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