An Adaptive LQG Approach to Nonlinear System Control

Željko M. Đurović[1] and Branko D. Kovačević[2]

Abstract - **The suboptimal controller design, based on LQG strategy, has been proposed in this paper. The controller is suitable for nonstationary and nonlinear multivariable plants. The key steps in the proposed method are corresponding linearization of the nonlinear plant model around the suitably chosen set of operating points, and design of nominal process trajectory. The feasibility of the approach is demonstrated through its application to 6 degree of freedom aircraft model.**

Keywords – **Adaptive control, suboptimal solution, LQG strategy, nonlinear systems.**

I. Introduction

Two groups of optimization based adaptive controllers to linear plants have drawn wide attention in recent years and have been widely studied in the literature. The first one makes use of the input-output representation of linear system, coupled with the minimization of a generalized output error variance [1-4]. The main advantage of such a type of controllers, named selftuning controllers includes the relative simplicity of their derivation and implementation. However, the performance index selected for this approach doesn't minimize the errors in the state trajectories, as may be required in some applications. Also, the global stability of the controlled system requires the inverse system to be stable, which may exclude some non-minimum phase systems. Furthermore, design of a such controller is rather complicated for multivariable plants. The second group makes use of the state space representation of the system coupled with the linear quadratic Gaussian (LQG) optimal control theory and sequential model parametrization technique [1,5,6]. The optimal adaptive control algorithms obtained have advantage of being stable, of being applicable to any finite dimensional controllable and observable system, and of providing with an effective control of the errors in the process trajectories.

Extension of the LQG approach to the control of nonlinear and nonstationary plants are proposed here. In contrast to the original LQG approach, these extensions provide for tracking of prespecified nominal trajectory. A numerical approach based on the strategy of predictive control and optimization under constraints has been proposed as a suitable method for the nominal trajectory generation. Another important step in this algorithm is a linearization of the nonlinear model under consideration. A new method for linearizing of nonlinear plant model has also been proposed in the paper. The feasibility of the proposed control strategy has been demonstrating through its application to a six degree of freedom (6DOF) aircraft model.

II. Problem Formulation

Consider the nonlinear system described by a state space model

$$dx(t)/dt = f\left(x(t),u(t)\right) \tag{1}$$

where x(t) is n-dimensional state vector and $u(t)$ is m-dimensional control signal. Let us suppose further, that

$$x(t) = x^0 + \Delta x(t)\ ,\ u(t) = u^0 + \Delta u(t) \tag{2}$$

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what corresponds to the translation of the coordinate system, adopting $(x^0,\ u^0)$ as the new origin. Then, the state space model takes the form

$$\frac{d\left(x^0 + \Delta x(t)\right)}{dt} = f\left(x^0 + \Delta x(t), u^0 + \Delta u(t)\right) \tag{3}$$

Taking into account that x^0 and u^0 are constant vectors, and expanding the nonlinear system characteristic f(·) into Taylor series approximation of the first order, one obtains

$$\begin{aligned}\frac{d\Delta x(t)}{dt} = f\left(x^0,u^0\right) + \left.\frac{\partial f(x,u)}{\partial x}\right|_{\substack{x=x^0\\u=u^0}} \Delta x(t)\\ + \left.\frac{\partial f(x,u)}{\partial u}\right|_{\substack{x=x^0\\u=u^0}} \Delta u(t)\end{aligned} \tag{4}$$

The proposed approach results into linearized system model

$$\frac{d\Delta x(t)}{dt} = \Phi\Delta x(t) + \Gamma\Delta u'(t) \tag{5}$$

where

$$\begin{aligned}\Phi = \left.\frac{\partial f(x,u)}{\partial x}\right|_{\substack{x=x^0\\u=u^0}};\Delta u'(t) = \begin{bmatrix}\Delta u(t)\\1\end{bmatrix};\\ \Gamma = \begin{bmatrix}\left.\frac{\partial f(x,u)}{\partial u}\right|_{\substack{x=x^0\\u=u^0}} & f\left(x^0,u^0\right)\end{bmatrix}.\end{aligned} \tag{6}$$

It should be noted that this specific linearization is not performed around the nominal trajectory, but around the fixed operating point in the $(n+m)^{th}$ dimensional space. This is more suitable from the numeric point of view, but the dimension of control signal vector is increased by one. The obtained linearized model is suitable for later application of LQG strategy, as is described in the next section.

The LQG strategy assumes a linear state space model

$$\begin{aligned}x(k+1) &= \Phi x(k) + \Gamma u(k) + v(k)\\ y(k) &= Hx(k) + e(k)\end{aligned} \tag{7}$$

where the matrices Φ,Γ and H are known, with v and e being zero-mean white Gaussian sequences with known covariances matrices

$$\begin{aligned}&E\left\{v(k)v^T(k)\right\} = R_1; E\left\{v(k)e^T(k)\right\} = R_{12};\\ &E\left\{e(k)e^T(k)\right\} = R_2\end{aligned} \tag{8}$$

The aim of control is to minimize the performance index

$$\begin{aligned}J = E\Bigg\{\sum_{k=0}^{N-1}\Big[x^T(k)Q_1x(k) + 2x^T(k)Q_{12}u(k)\\ + u^T(k)Q_2u(k)\Big] + x^T(N)Q_0x^T(N)\Bigg\}\end{aligned} \tag{9}$$

where Q_0, Q_1, Q_{12} and Q_2 are given weighting matrices and NT represents the optimization horizon, with T being the sampling period. The control obtained by minimizing (9) will be refered to as the LQG state feedback control and is given by [1,5,6]

$$u(k) = -L(x)x(k) \tag{10}$$

where:

$$L(k) = \left(Q_2 + \Gamma^T S(k+1)\Gamma\right)^{-1}\left(\Gamma^T S(k+1)\Phi + Q_{12}^T\right) \tag{11}$$

The matrix sequence S(k) is a solution of the discrete Riccati equation:

$$\begin{aligned}S(k) = \left[\Phi - \Gamma L(k)\right]^T S(k+1)\left[\Phi - \Gamma L(k)\right]\\ + Q_1 + L^T(k)Q_2L(k)\end{aligned} \tag{12}$$

with the initial condition $S(N)=Q_0$. For the time-invariant model (Φ,Γ,H) the Riccati equation (12) has to be solved in advance, and the corresponding gain sequence (11) can be stored and used later for control purposes.

Although optimal, LQG strategy cannot be applied directly in the case of nonlinear and/or nonstationar plants. Therefore, a suitable modification of this strategy to nonlinear or nonstationary plants control is going to be proposed in the following section.

III. Adaptive LQG controller

The method discussed above may be extended so to design an estimated state feedback controller for non-linear and non-stationary systems with non-zero reference signal. The scheme for such implementation of the LQG controller is shown in figure 1. Here, $u_{ref}(k)$ and $y_{ref}(k)$ represent a given time-varying deterministic reference, or nominal trajectory, while u(k) and y(k) are deviations from the nominal signals $u_{ref}(k)$ and $y_{ref}(k)$, respectively.

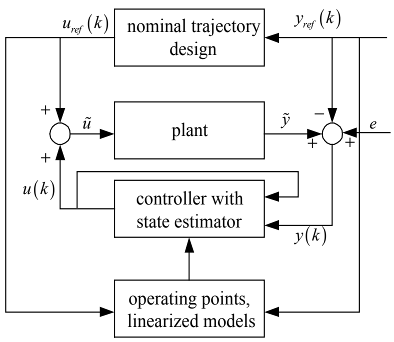

Fig. 1: Adaptive LQG controller

The signal u(k), generated by controller, represents a correction around the nominal, or reference, control signal. Thus, the adopted structure of the control system enables one to track the reference trajectory of the desired form. Furthermore, since the adaptive LQG approach uses the linearized models depending on actual operating points, it can also be applied to nonlinear systems. The choice of operating points should be done in such way to cover characteristic nonlinear regimes of the system concerned. In this way, various linear models will describe the behaviour of the nonlinear system in the vicinity of the chosen set of operating points.

In practice, the reference trajectory is usually obtained either by developing a complex nonlinear model of the system in question or by simulation under some reasonable operating conditions. Adopted here is numerical approach based on the strategy of predictive control and optimization under constraints ($u_{min} \leq u_{ref} \leq u_{max}$, where bounds u_{min} and u_{max} have to be determined on the basis of known process properties).

Let y_{ref}^{nom} represents the desired nominal trajectory, then the nominal control signal u_{ref}, at stage *k*, can be calculated to minimize the deflection from the desired nominal trajectory, i.e.

$$u_{ref}(kT) = \arg\left\{\min_{u_{ref}}\left(y_{ref}\{(k+N)T\} - y_{ref}^{nom}\{(k+N)T\}\right)^2\right\} \tag{13}$$

where $y_{ref}\{(k+N)T\}$ denotes the y_{ref}-coordinate at the instant $(k+N)T$. The sequence $y_{ref}\{(k+N)T\}$ is obtained by solving the corresponding nonlinear state-space equation of the system concerned under the condition that the control signal $u_{ref}(t)$ is constant over *N* consequitive sampling periods *k*, *k*+1, ...,*k*+*N*-1, respectively. Figure 2 represents the flowchart of the proposed algorithm for calculating the desired reference control signal under which the desired reference output trajectory will be achieved. One can use, for example, the Runge-Kutta method of the fourth or fifth order for obtaining the state trajectory and the response of a nonlinear system model in question. Furthermore, a gradient-type procedure for solving the optimization problem (13) can also be used. Particularly, the Nelder-Mead direct search method represents a good procedure to be used for this purpose [7].

Prediction horizon *N* represents a free parameter, which has to be adopted in advance. The choice of *N* represents a compromise between two opposite requirements concerning the allowable values and dynamics of the reference control input u_{ref} and the corresponding admissible errors in tracking the prespecified reference trajectory. A smaller *N* will result into the control signal u_{ref} very close to the prespecified bounds to minimize as fast as possible the deflections from the nominal trajectory. On the other hand, higher *N* will result in the control signal with smaller dynamics, which is not influenced too much by the given control bounds. However, this will yield rather large deflections from the given nominal trajectory, as is shown by simulation in the next section.

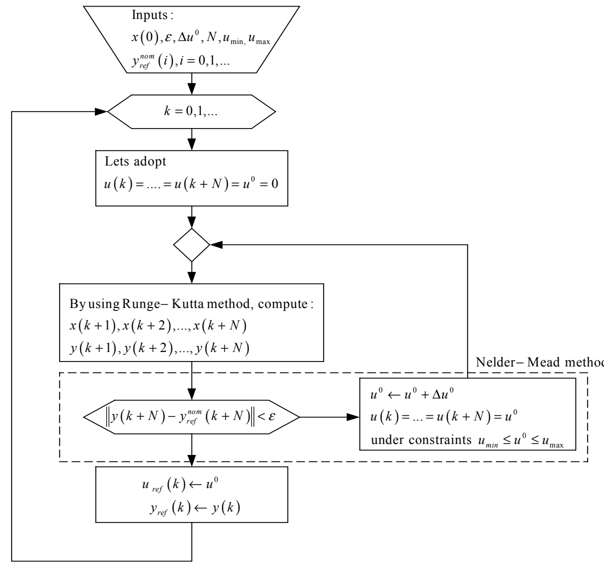

Fig. 2. Calculation of the nominal control signal

Finally, let us note that the proposed strategy requires to combine the LQG controller with an estimator, since the whole system state vector is not measurable. Therefore, the proper estimator has to be designed using the Kalman estimator and the corresponding linearized models [6,8]. In a such estimator design the matrices R_1, R_{12} and R_2 in (8), as well as *H* in (7), are assumed to be known apriori. However, this step is omitted due to the space limitation of the article.

IV. Simulated example

To demonstrate the feasibility of the approach, consider the example of an aircraft control around the state reference trajectory. An aircraft motion is described by 12 standard nonlinear differential equations given below, known in the literature as the six degrees-of-freedom (6DOF) flight model [9]

a) Dynamical translatory equations in body axes:

$$\frac{du}{dt} = -\frac{\rho V^2}{2}[C_D\cos(\alpha)\cos(\beta) + C_Y\cos(\alpha)\sin(\beta) + C_L\sin(\alpha)] + T/m - g\sin(\Theta) - qw + rv \tag{14}$$

$$\frac{dv}{dt} = -\frac{\rho V^2}{2}\left[-C_D\sin(\beta) + C_Y\cos(\beta)\right] + g\sin(\phi)\cos(\Theta) - ru + pw \tag{15}$$

$$\frac{dw}{dt} = -\frac{\rho V^2}{2}[-C_D\sin(\alpha)\cos(\beta) - C_Y\sin(\alpha)\sin(\beta) + C_L\cos(\alpha)] - g\cos(\Theta)\sin(\phi) - pv + qu \tag{16}$$

b) Dynamical angular equations in body axis:

$$\frac{dp}{dt} = C_l \frac{1}{2} \rho V^2 Sd / A \qquad (17)$$

$$\frac{dq}{dt} = \left(C_m \frac{1}{2} \rho V^2 Sd / B + (C - A) rp \right) / B \qquad (18)$$

$$\frac{dr}{dt} = C_n \frac{1}{2} \rho V^2 Sd / C + \left((A - B) pq \right) / C \qquad (19)$$

c) Kinematical (Euler) equations

$$\frac{d\phi}{dt} = p + \left(r\cos(\phi) + q\sin(\phi) \right) \tan(\Theta) \qquad (20)$$

$$\frac{d\Theta}{dt} = q\cos(\phi) - r\sin(\phi) \qquad (21)$$

$$\frac{d\psi}{dt} = \left(r\cos(\phi) + q\sin(\phi) \right) / \cos(\Theta) \qquad (22)$$

d) Transformation of ground speed from body to Earth coordinates:

$$\begin{aligned} \frac{dX}{dt} &= u\cos(\Theta)\cos(\psi) \\ &+ v(\sin(\psi)\sin(\Theta)\cos(\psi) - \cos(\phi)\sin(\psi)) \\ &+ w\left(\cos(\phi)\sin(\Theta)\cos(\psi) + \sin(\phi)\sin(\psi)\right) \end{aligned} \qquad (23)$$

$$\begin{aligned} \frac{dY}{dt} &= u\cos(\Theta)\sin(\psi) \\ &+ v\left(\sin(\phi)\sin(\Theta)\sin(\psi) + \cos(\phi)\cos(\psi)\right) \\ &+ w\left(\cos(\phi)\sin(\Theta)\sin(\psi) - \sin(\phi)\cos(\psi)\right) \end{aligned} \qquad (24)$$

$$\frac{dZ}{dt} = -u\sin(\Theta) + v\sin(\phi)\cos(\Theta) + w\cos(\phi)\cos(\Theta) \qquad (25)$$

Here, ρ is air density, S is the reference of the cross-sectional area, d is body diameter, m is aircraft mass, g is Earth acceleration and T is thrust moment. The absolute aircraft speed V, angle of attack α and angle of sideslip β are given by:

$$V = \sqrt{u^2 + v^2 + w^2} \qquad (26)$$

$$\alpha = \tan^{-1}(w/u)\ ;\ \beta = \tan^{-1}(v/u) \qquad (27)$$

The parameters A, B and C are time varying, since they are dependent on the aircraft mass and drag force, while the aerodynamic coefficients C_D, C_L, C_Y, C_m, C_n and C_l represent the functions of angles α and β, the angular velocities p, q, r and the angles δ_A, δ_E and δ_R. The functional relations between parameters characterizing the 6DOF model are given in [9]. The last three angles are the input variables satisfying the boundary condition $|\delta| \leq 0.2 rad$. Thus, the system of nonlinear differential equations can be represented in the nonlinear state space form (1), where the state vector is defined by

$$x(t) = [u, v, w, p, q, r, \phi, \psi, \theta, X, Y, Z]^T$$

and $u(t) \in R^3$ is the control vector given by

$$u(t) = [\delta_A, \delta_E, \delta_R]^T,$$

where the three elements stand for aileron, elevator and rudder, respectively. In addition, f(·)is the corresponding 12th dimensional nonlinear vector function, describing the nonlinear system dynamics. The reference trajectory, presented in figure 3, is specified in the (X,Z) plane taking into account the flight conditions and real capabilities of the aircraft.

Additionally, it is necessary to calculate the nominal control $u_{nom}=[\delta_A^{nom}, \delta_E^{nom}, \delta_R^{nom}]$ across the given reference trajectory Z^{nom}. However, since the trajectory is defined in the vertical plane, it is natural to choose $\delta_R^{nom}=\delta_A^{nom}=0$. Thus, the 12th order nonlinear system (1) with the one input $u_{nom}(t)= \delta_E^{nom}(t)$ and the one output $y_{nom}(t)= Z^{nom}(t)$ is defined.

The prediction horizon N represents the free parameter that has to be chosen in advance. Figures 4 and 5 depict the control signal for different values of N, while figure 6 shows the corresponding deflections from the nominal trajectory. As a result of this brief experimental analysis, it is chosen as N=4. The calculated nominal control for the chosen N is given in figure 7.

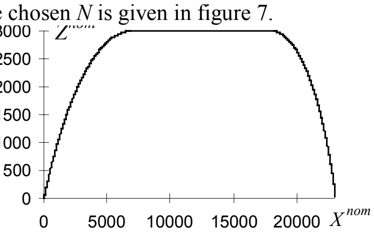

Figure 3. Nominal trajectory

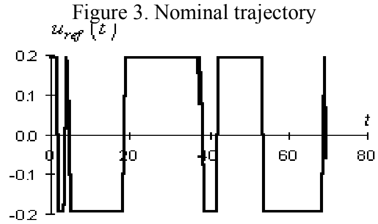

Figure 4. Nominal control signal for N=2

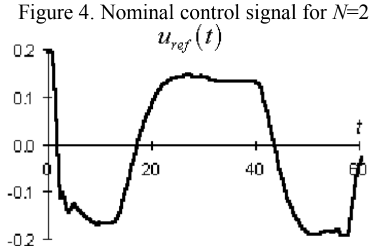

Figure 5: Nominal control signal for N=8

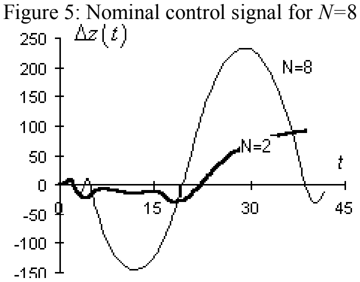

Figure 6: Deflections from the nominal trajectories for different values of N

The following important step in the controller design procedure was to make a proper linearization of nonlinear plant model. Based on the equations (3-6) the corresponding matrices Φ,Γ have been obtained. In order to check the quality of linearization, we compared the responses of nonlinear plant and linearized model, when the nominal input signal has been applied. The figures 8 and 9 depict the responses of the 5th and 12th state coordinates for both cases. Based on these results one can conclude that the linearized model approximates the nonlinear plant properly. It should be also noted that during the chosen simulation the aircraft was ramping with rapid mass change and this, in turn, causes the significant plant dynamic changes.

Figure 10 represents the change of dominant time constant of the linearized model during the nominal flight. This behaviour suggests the number and position of the operating points at the nominal trajectory, that have to be used to cover the changes of plant dynamics during the flight. These operating points have to be chosen carefully, because the linearization process, and the controller design, as well, are strongly based on them. Namely, the nonlinear plant is going to be described by a set of linearized models sequence. The number of these models should be high enough to approximate the nonlinear plant behaviour properly, but, on the other hand, too large number of operating points, and corresponding linearized models, increases the numerical complexity significantly. So, some compromise between these two requirements has to be done. Starting from figures 3 and 10, six operating points have been chosen. Thus, the ramp of the flight has been covered by three operating points, the horizontal part by one, and the last portion of the trajectory by two operating points.

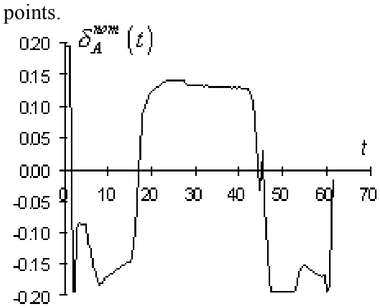

Figure 7. Nominal control for N=4

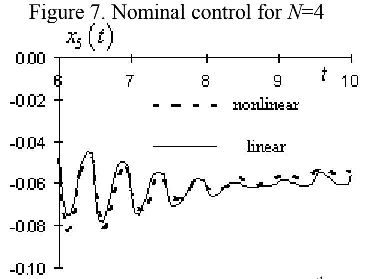

Figure 8. Response to the nominal control of the 5th state coordinate

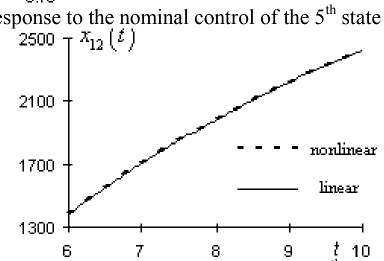

Figure 9. Response to the nominal control of the 12th state coordinate

The succesful application of LQG strategy also depends on the choice of the weighting matrices Q_0, Q_1, Q_2 and Q_{12}. In practice, these matrices may be chosen by simulation. As a result of such analysis, they are chosen as

$$Q_0 = Q_1 = I_{12}, Q_2 = 3500\times diag\{1,1,0.8\}, Q_{12} = 0_{12\times 3}$$

where I and O are identity and zero matrices, of the corresponding orders, respectively.

Simulation of the closed-loop system is performed in the presence of the additive measurement noise *e*, representing white, zero-mean Gaussian sequence with variance R_2=50.

Figures 11 and 12 show the control signal *u(k)* generated by LQG control law and the deflection of the aircraft altitude from the nominal trajectory.

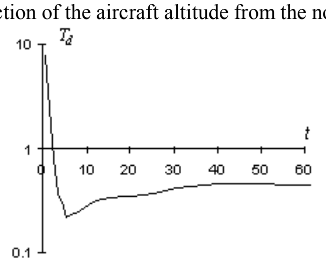

Figure 10. Time constant history of the linearized model during the nominal flight

It can be concluded that the control signal is rather small, so that the large control signal $\tilde{u}$ (fig. 1) satisfies the adopted constraints. It should be noted that this goal can be achieved by choosing properly the weighting matrix Q_2.

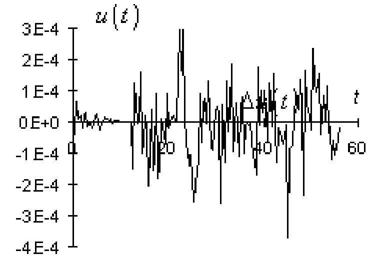

Figure 11. Output of LQG controller

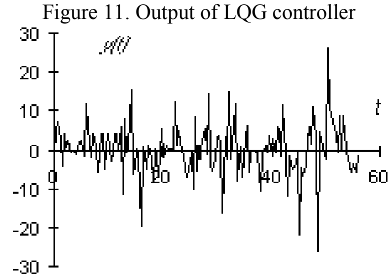

Figure 12. Deflection of the real altitude from the nominal trajectory

V. CONCLUSION

The form of adaptive LQG design for nonlinear and nonstationary plants control has been proposed here. In contrast to the known LQG approach from the literature, the proposed approach enables tracking of a desired non-zero reference trajectory. The key design steps are the generation of reference trajectory by using numerical optimization with constraints, a specific linearization of the nonlinear plant model around the fixed operating points, the choice of operating points in accordance to the plant dynamics changes and the choice of weighting matrices of the classical LQG controller. All these steps are analyzed in details in the paper. The time constant history of the linearized model and the nominal reference trajectory were used as a criterion for choosing the operating points. The feasibility of the proposed approach has been demonstrated through the example of aircraft control around the prespecified reference trajectory. The obtained results have shown that the proposed controller may represent an efficient tool for tracking arbitrary desired trajectory in the case of nonlinear and nonstationary system dynamics, as well as, in the presence of measurement noise.

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