Synchronization of Modulated Two-Phase Harmonic Oscillator Employing Discrete-Time Quasi-Sliding Mode

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Abstract – **This paper explores two-phase harmonic oscillator synchronization with referent oscillator which amplitude and phase are modulated by two arbitrary smooth functions, respectively. Employing discrete-time variable structure control system, which organize quasi-sliding mode along intersection of two appropriate sliding surfaces, pursues synchronization between modulated and controlled oscillator. Discrete-time relay-type variable structure controllers are proposed and quasi-sliding reaching and existence conditions are derived, verified by experimental results.**

Keywords –**Synchronization, Modulated harmonic oscillator, Variable structure control system, Quasi-sliding mode.**

I. Introduction

Considerable application of harmonic oscillators in many fields of electrical engineering, such as, power converters, supply systems, laboratory research, system examinations, electric drive control, instrumentation etc., emphasizes them as fundamental devices. Basic demands for oscillator reliability are: high quality amplitude and frequency regulation and stability, short oscillation build-up time and low-level of harmonic distortions. An adequate approach to oscillator synthesis meeting all above-mentioned requirements is an introduction of variable structure control system (VSCS) leading to a sliding motion on a desired surface [1], originally proposed by Sira-Ramirez in Van der Pol oscillator control, [2]. It has been shown that sliding mode along certain elliptical trajectory in system state space provides an ideally sinusoidal response with desired amplitude. Following this concept a systematic approach to synthesis of two and three-phase harmonic as well as relaxation oscillators, has been proposed in [3,4].

Further research has been directed to regulation of both amplitude and phase of harmonic oscillators using sliding mode control. It has been shown in [3,5] that control system enhancement by introduction of additional control signal and related sliding surface responsible for phase regulation provides absolute oscillation control. Desired amplitude and phase is obtained by eventual sliding mode on the intersection of two appropriate nonlinear sliding surfaces. This feature is essential to oscillator synchronization and has been employed in two and three-phase harmonic oscillator synchronization [6,7]. All desirable properties characterizing sliding mode is preserved by this control strategy resulting in high-quality synchronization in finite time featuring high accuracy of both amplitude and phase, invariance to parameter perturbations and external disturbances and great harmonic clearness, [7].

This paper investigates a further extended synchronization problem, i.e. synchronization with modulated two-phase harmonic oscillator. Oscillation amplitude and phase are modulated by two arbitrary smooth functions, respectively. Sliding mode control approach is employed in handling this synchronization task. Discrete-time relay-type VSCS is suggested which organizes quasi-sliding motion along the intersection of two sliding surfaces, individually providing amplitude and phase synchronization.

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II. Control System Description

In order to provide oscillation amplitude and phase regulation, which is essential to synchronization with a referent oscillator, a conservative two-phase harmonic oscillator

$$\begin{aligned}\dot{x}_1 &= \omega_0 x_2,\\ \dot{x}_2 &= -\omega_0 x_1,\end{aligned}\tag{1}$$

which amplitude and initial phase depend on initial conditions whereas oscillation frequency is determined by system parameter ω_0, has been enhanced in [6] by introduction of control signals in the following manner:

$$\begin{aligned}\dot{x}_1 &= \omega_0 x_2 + x_2 u_2 + x_1 u_1,\\ \dot{x}_2 &= -\omega_0 x_1 - x_1 u_2 + x_2 u_1.\end{aligned}\tag{2}$$

Influence of the control signals on the oscillation amplitude and phase is more explicit in polar coordinate system. Using coordinate transformation

$$r^2 = x_1^2 + x_2^2, \quad \theta = \mathrm{arctg}\frac{x_1}{x_2},\tag{3}$$

controlled oscillatory system (2) is transformed to

$$\begin{aligned}\dot{r} &= r u_1,\\ \dot{\theta} &= \omega_0 + u_2.\end{aligned}\tag{4}$$

It is obvious that controlled system (2) provides completely decoupled amplitude and phase control, indicating that the proposed structure is convenient for synchronization tasks.

A microprocessor based realization of the system (2) allows synthesis of the nonlinear control part using digital hardware. Therefore, initial continuous-time model (2) is represented in vector form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{c}, \quad \mathbf{A} = \begin{bmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}, \quad (5)$$

where $\mathbf{x}^{\mathrm{T}} = [x_1, x_2]$ and $\mathbf{c}^{\mathrm{T}} = [c_1, c_2]$. Vector $\mathbf{c}$ is evaluated in discrete time instants, remaining constant during sampling period $\mathbf{c}(t) = \mathbf{c}(kT)$, $kT \le t < (k+1)T$, $k \in N_0$, according to

$$\begin{aligned} c_1(k) &= x_1(k)u_1(k) + x_2(k)u_2(k), \\ c_2(k) &= x_2(k)u_1(k) - x_1(k)u_2(k). \end{aligned} \quad (6)$$

Oscillator discrete-time model in polar representation, which is necessary for system analysis, is formed using Eq. (3) and discrete-time state space model

$$\mathbf{x}(k+1) = \mathbf{E}(T)\mathbf{x}(k) + \mathbf{F}(T)\mathbf{c}(k), \quad (7)$$

obtained performing standard discretisation method. Time discretisation process has considerably extended model complexity and violated decupled control property, making system analysis more complicated. Since very small sampling period T is feasible nowadays due to powerful microcontrollers, it is quite reasonable to use model approximation in order to simplify system analysis. Model approximation is implemented using Taylor polynomial. A differentiable function $f(x)$ in a vicinity of point a can be approximated by first order Taylor polynomial

$$T_1(x) = f(a) + f'(a)(x-a), \quad (8)$$

committing an approximation error which is defined in Lagrange form as

$$R_1(x) = \frac{f''[a + \eta(x-a)]}{2!}(x-a)^2, \quad 0 < \eta < 1. \quad (9)$$

In the case of study, approximation of functions representing oscillator discrete-time polar model $f_1(T) = r^2(k+1)$ and $f_2(T) = \theta(k+1)$ is performed in a vicinity of point $T = 0$. The resulting approximated model is obtained according to Eq. (8) as

$$\begin{aligned} r^2(k+1) &= (1 + 2u_1(k)T)r^2(k), \\ \theta(k+1) &= \theta(k) + (\omega_0 + u_2(k))T. \end{aligned} \quad (10)$$

Using Eq. (9), modeling error becomes

$$R_{i1}(T) = \frac{f_i''(\eta T)}{2!}T^2, \quad 0 < \eta < 1, \quad i = 1,2, \quad (11)$$

indicating that if small sampling period T is applied, modeling error is negligible and the discrete-time model (10) is valid. Model decoupling in Eq. (10), as in continuous-time model (4), is here an approximation result.

Control objective is to synchronize two-phase harmonic oscillator (10) with a referent oscillator, which amplitude and phase are modulated. Output signals of a referent oscillator may be described as

$$\begin{aligned} x_{r1}(t) &= r_r(t)\sin(\theta_r(t)), \\ x_{r2}(t) &= r_r(t)\cos(\theta_r(t)), \end{aligned} \quad (12)$$

where oscillation amplitude $r_r(t)$ and phase $\theta_r(t)$ are arbitrary smooth functions. In order to attain amplitude and phase matching, let the switching functions be

$$s_1(k) = r_r^2(k) - r^2(k), \quad s_2(k) = \theta_r(k) - \theta(k). \quad (13)$$

Obviously, if its possible to ensure an eventual sliding mode on the intersection of sliding surfaces $s_1 = 0$ and $s_2 = 0$, synchronization requirements $r(t) = r_r(t)$ and $\theta(t) = \theta_r(t)$ will be met. Oscillation amplitude and phase of referent as well as controlled oscillator can be evaluated according to transformation (3) using sampled outputs. Now, control laws $u_1(k)$ and $u_2(k)$ should be determined which provide the desired sliding motion, regarding the fact that control design can be performed separately. However, digital realization of control algorithm, due to inherent time delay caused by time discretisation process, may allow only existence of quasi-sliding mode [8], where system trajectories are constraint in some small bounded vicinity of sliding surface. This reduces system performance quality.

III. AMPLITUDE SYNCHRONIZATION

Quasi-sliding motion implies that system trajectory is in some bounded vicinity of sliding surface $s_1 = 0$, which is called quasi-sliding region S_1. Therefore quasi-sliding existence condition requires quasi-sliding region S_1 to be an invariant set, i.e. every system motion, described by Eq. (10), which has arrived into S_1 in some time instant $t_0 = k_0 T$, remains in the region S_1 for every $k > k_0$, $k, k_0 \in N$.

According to Eqs. (10) and (13), system motion toward the first sliding surface is

$$s_1(k+1) = s_1(k) + \Delta a(k+1) - 2Tr^2(k)u_1(k), \quad (14)$$

where $\Delta a(k+1) = r_r^{\,2}(k+1) - r_r^{\,2}(k)$. In order to conduct system motion (14) towards the sliding surface by means of control signal, prediction of reference amplitude $\Delta a(k+1)$ is needed. This information is unavailable so control signal $u_1(k)$ is chosen as reley-type function in the form

$$u_1(k) = \left(\alpha_0 + |\Delta a(k)| / 2Tr^2(k)\right)\mathrm{sgn}(s_1(k)), \quad (15)$$

under constraint $r(k) \ne 0$, $\forall k \in N$. Since only initial moment is critical, constraint is reduces to $r(0) \ne 0$, implying simultaneously nonzero initial conditions which is not too restrictive. In case of Eq. (15), (14) becomes

$$\begin{aligned} s_1(k+1) = s_1(k) &+ \Delta a(k+1) - |\Delta a(k)|\mathrm{sgn}(s_1(k)) \\ &-2Tr^2(k)\alpha_0 \,\mathrm{sgn}(s_1(k)). \end{aligned} \quad (16)$$

Suppose that $s_1(k) = 0_+$, that is $s_1(k) > 0$ and system trajectory is infinitely close to the surface $s_1 = 0$. Quasi-sliding existence requires system trajectory to cross over the sliding surface during the next time interval. Hence, Eq. (16) yields

$$s_1(k+1) = \Delta a(k+1) - |\Delta a(k)| - 2Tr^2(k)\alpha_0 < 0. \quad (17)$$

Fulfillment of inequality (17) requires

$$\alpha_0 > d_1(k) / 2Tr^2(k), \quad (18)$$

where $d_1(k) = \Delta a(k+1) - |\Delta a(k)|$.

Suppose now that $s_1(k) = 0_-$. Respecting the requirement of crossing the sliding surface during the next time interval, condition $s_1(k+1) > 0$, according to Eq. (16), becomes

$$\alpha_0 > -d_2(k) / 2Tr^2(k). \quad (19)$$

were $d_2(k) = \Delta a(k+1) + |\Delta a(k)|$.

Differences in Eqs. (18) and (19) may be treated as disturbances, which are bounded with limits $|d_i(k)| \le \mu_i$, $i = 1,2$, in case of smooth amplitude $r_r(t)$ reference. Inequality satisfying both cases constitutes quasi-sliding existence condition

$$\alpha_0 > \max(\mu_1, \mu_2)/2Tr_{\min}^2 . \qquad (20)$$

For small sampling period it can be regarded $\Delta a(k+1) \approx \Delta a(k)$, which demonstrates that in practical realizations controller parameter α_0 should be set as a small value.

Quasi sliding region S_1 represents the range of function $s_1(k+1)$ variation and may be depicted as

$$d_1(k) - 2Tr_{\max}^2\alpha_0 < s_1(k+1) < d_2(k) + 2Tr_{\max}^2\alpha_0 . \qquad (21)$$

IV. Phase Synchronization

The same reasoning is employed in analysis and design of the system motion toward and within the quasi-sliding region S_2, defined in the same way as S_1.

System motion toward the surface $s_2 = 0$ is defined, using Eqs. (10) and (13), as

$$s_2(k+1) = s_2(k) + \Delta p(k+1) - (\omega_0 + u_2(k))T , \qquad (22)$$

where $\Delta p(k+1) = \theta_r(k+1) - \theta_r(k)$. Control signal $u_2(k)$ is chosen as relay-type function

$$u_2(k) = (\beta_0 + |\Delta p(k)|/T)\operatorname{sgn}(s_2(k)), \qquad (23)$$

from the same reasons as in amplitude synchronization, transforming Eq. (22) into

$$\begin{aligned} s_2(k+1) = s_2(k) + \Delta p(k+1) - |\Delta p(k)|\operatorname{sgn}(s_2(k)) \\ -\beta_0 T\operatorname{sgn}(s_2(k)) - \omega_0 T. \end{aligned} \qquad (24)$$

If $s_2(k) = 0_+$, requirement for system trajectory to cross the sliding surface during next sampling period $s_2(k+1) < 0$ yields condition

$$\beta_0 > d_3(k)/T - \omega_0 , \qquad (25)$$

where $d_3(k) = \Delta p(k+1) - |\Delta p(k)|$. In case of $s_2(k) = 0_-$, system trajectory will cross the sliding surface during the next sampling interval if $s_2(k+1) > 0$, imposing

$$\beta_0 > -d_4(k)/T + \omega_0 , \qquad (26)$$

where $d_4(k) = \Delta p(k+1) + |\Delta p(k)|$. Disturbances $d_i(k)$, $i = 3,4$, for smooth phase reference $\theta_r(t)$ are bounded with limits $|d_i(k)| \le \mu_i$, $i = 3,4$. Finally, quasi-sliding existence condition, as a conjunction of inequalities (25) and (26), is described as

$$\beta_0 > \max(\mu_3/T - \omega_0, \mu_4/T + \omega_0). \qquad (27)$$

Small sampling period allows further simplification of condition (27) because of presumption $\Delta p(k+1) \approx \Delta p(k)$.

Variation of function $s_2(k+1)$ defines quasi-sliding region S_2, which can be, according to Eq. (24), depicted as

$$d_3(k) - \omega_0 T - \beta_0 T < s_2(k+1) < d_4(k) - \omega_0 T + \beta_0 T . \quad (28)$$

Limits of disturbances $d_i(k)$, $i = 1 \div 4$, are proportional to period T affecting the width of regions S_1 and S_2, which determine amplitude and phase tracking errors. To gain high system accuracy, sampling period T should be as small as possible.

V. Reaching of the Quasi-Sliding Regions

Arrival into quasi-sliding regions S_i, $i = 1,2$ from an arbitrary initial point in finite number of sampling intervals, will be proved simultaneously using contradiction principle. Assume that system trajectories never reach regions S_i. Hence, if $s_i(0) > 0$, then $s_i(k) > 0$, $i = 1,2$, $\forall k \in N$. Evolving recurrent relations (16) and (24), $s_i(k)$ with respect to initial value can be expressed as

$$s_1(k) = s_1(0) - \sum_{i=0}^{k-1}\left(2Tr^2(i)\alpha_0 - d_1(i)\right), \qquad (29)$$

$$s_2(k) = s_2(0) - \frac{1}{T}\sum_{i=0}^{k-1}(\omega_0 + \beta_0 - d_3(i)/T). \qquad (30)$$

Series with common elements (29) and (30) are monotonously decreasing and bounded in case of conditions (18) and (25), respectively. According to theorem on monotonous series convergence, limit values $\lim_{k\to\infty} s_i(k) = s_{i\infty}$ must exist. Using this, Eqs. (29) and (30) can be rewritten as

$$\sum_{i=0}^{\infty}\left(2Tr^2(i)\alpha_0 - d_1(i)\right) = s_1(0) - s_{1\infty}, \qquad (31)$$

$$\sum_{i=0}^{\infty}(\omega_0 + \beta_0 - d_3(i)/T) = T(s_2(0) - s_{2\infty}). \qquad (32)$$

Obviously, arrays (31) and (32) are convergent and according to Cauchy array convergence criterion their common elements must tend to zero. Hence, $\alpha_0 = \frac{1}{2T}\lim_{i\to\infty}\left(d_1(i)/r^2(i)\right)$ and $\beta_0 = \frac{1}{T}\lim_{i\to\infty} d_3(i) - \omega_0$, which are in contradiction with the derived conditions (18) and (25), respectively. Therefore, the assumption made above is incorrect and system trajectories will reach quasi-sliding regions S_i from an arbitrary initial point in finite number of sampling intervals. Consequently, conditions (20) and (27) are proved to be reaching conditions as well.

VI. Experimental Results

The designed discrete-time VSCS ensuring synchronization has been examined experimentally using two-phase harmonic oscillator controlled by a microprocessor. Two-phase harmonic oscillator (5) with natural frequency $\omega_0 = 6\,\text{rad/s}$ has been created by means of operational amplifiers. Two-phase referent signals, with modulated amplitude and phase (12), have been generated by the microprocessor. In the case of study amplitude and phase modulation has been chosen as

$$r_r(t) = r_{r0} + \Delta r_r \sin(\omega_a t),$$
$$\theta_r(t) = \theta_{r0} + \omega_r t + \Delta\theta_r \sin(\omega_p t), \quad (33)$$

where signal parameters have been selected as follows:

$r_{r0} = 0.9\,\text{V}$, $\Delta r_r = 0.4\,\text{V}$, $\omega_a = 3.14\,\text{rad/s}$;

$\theta_{r0} = 0.5\,\text{rad}$, $\omega_r = 7.5\,\text{rad/s}$, $\Delta\theta_r = 1.8\,\text{rad}$, $\omega_p = 2\,\text{rad/s}$.

Discrete-time VS controllers has been designed according to Eqs. (6), (15), and (23). Controller parameters have been set to $\alpha_0 = 1$ and $\beta_0 = -1.5$, regarding derived conditions (20) and (27). Sampling period has been $T = 250\,\mu s$.

Switching functions $s_1(t)$ and $s_2(t)$ are shown in figure 1. System trajectory first reaches quasi-sliding region S_1, and remaining within reaches region S_2. Thus, an eventual quasi-sliding mode occurs in a vicinity of the intersection of sliding surfaces $s_1 = 0$ and $s_2 = 0$. Dimensions of the quasi-sliding regions determine amplitude and phase error. Data analysis has shown that amplitude relative error is less than 0.6 %, whereas phase absolute error is less then 0.02 rad. Figure 2 shows referent signals $x_{r1}(t)$, $x_{r2}(t)$ and state coordinates $x_1(t)$, $x_2(t)$. It can be noticed that controlled oscillator output signals are successfully synchronized with the referent signals. Control signals are shown in figure 3.

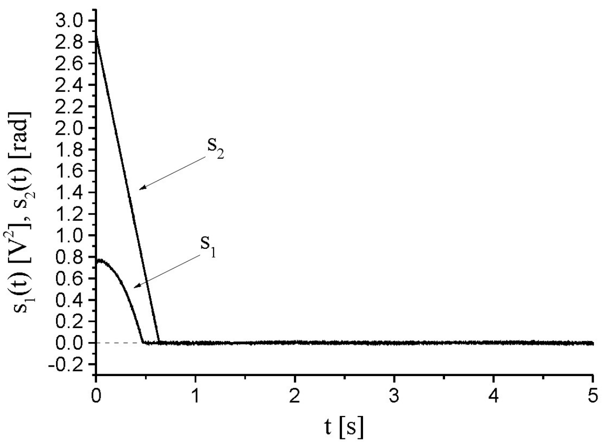

Fig. 1. Switching functions

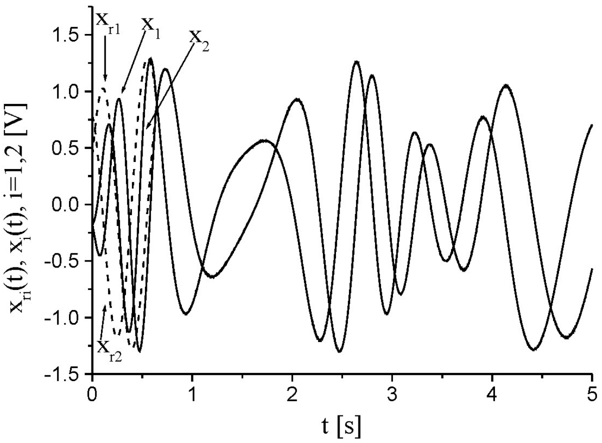

Fig. 2. Output signals

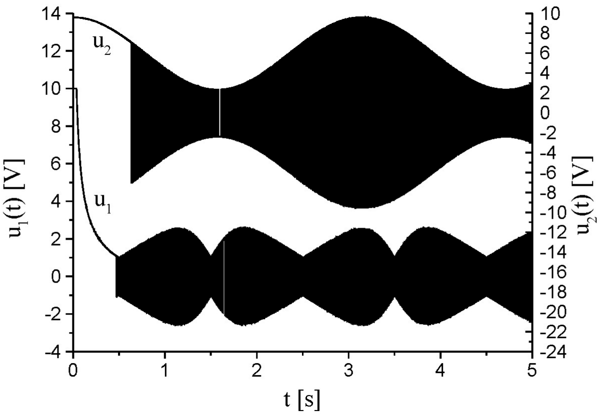

Fig. 3. Control signals

VII. Conclusion

Synchronization of two-phase harmonic oscillator with modulated amplitude and phase has been investigated in this paper. It has been shown that proposed discrete-time reley-type VSCS, leading to eventual quasi-sliding regime in a small bounded vicinity of intersection of two sliding surfaces, successfully provides high quality synchronization. Small sampling time and proper choice of controller parameters significantly narrows quasi-sliding regions resulting in high accuracy of both amplitude and phase. Experimental results support the presented sliding mode based synchronization method and verify the designed discrete-time variable structure controllers.

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