Direct Design of Transitional Butterworth-Chebyshev IIR Filters

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Abstract – **In this paper the procedure for direct design of selective IIR digital filters has been described. These filters can not be obtained using transformations from analogous domain. The complete procedure for determining of coefficients of the magnitude squared characteristics has been presented. Slope of the amplitude characteristics has been also investigated. These filters provide much sharper cutoff slope characteristics, especially in the design of narrow band lowpass filters.**

Keywords – **IIR filters, Transitional filters, Minimax amplitude characteristics, Cutoff slope**

I. Introduction

Digital filters are playing an important and increasing role in design of modern telecommunication systems. Recursive digital filters are mostly used in design of selective amplitude characteristics. Transfer function of these filters can be obtained by indirect and direct methods. Indirect methods are well known, but it is not possible to design all kind of filters. In this paper the direct design of transitional Butterworth-Chebyshev digital filters with selective amplitude characteristics will be described.

The direct design of transitional Butterworth-Chebyshev digital filters with all zeroes at the origin has been described in [1]. The procedure for introducing single or multiple zero pairs on the unit circle directly in z domain has been described in [2]. By this way it is possible to obtain more selective amplitude characteristic. The introducing of zeros on the unit circle results in loosing of mini-max characteristic of the attenuation in the passband.

In order to improve again selectivity, the procedure for direct design of transitional filters with equal pass-band ripples, where characteristic function is obtained using simple iterative procedure, has been proposed in this paper. Obtained filters have increased selectivity if we compare them with those described in [2]. These filters can not be obtained using transformations from analogous domain.

II. Approximation

Magnitude squared functions of a transitional Butterworth-Chebyshev filter with m pairs of identical zeros at $e^{\pm j\omega_z T}$ is given by

$$|H_{n,k,m}(e^{j\omega T})|^2 = \frac{1}{1+\varepsilon^2 K_n^2(x)}, \tag{1}$$

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where ε is a real constant, x is a frequency variable in ωT suitable for low-pass filtering given by

$$x = \frac{\sin\dfrac{\omega T}{2}}{\sin\dfrac{\omega_c T}{2}} \tag{2}$$

and x_z is multiple zero on the unit circle

$$x_z = \frac{\sin\dfrac{\omega_z T}{2}}{\sin\dfrac{\omega_c T}{2}}, \tag{3}$$

where $\omega_c T$ is cutoff frequency and $\omega_z T > \omega_c T$ and $n \geq 2m$ where parameter n is $n = k+l$ and m is a number of multiple zero pairs on the unit circle at $w_z T$.

Characteristic function $K_n(x)$ is a rational function and can be written as

$$K_n(x) = P_n(x)\Big(\frac{x_z^2-1}{x^2-x_z^2}\Big)^m, \tag{4}$$

where $P_n(x)$ is a real polynomial equal to

$$P_n(x) = x^l(a_0 + a_2x^2 + \cdots + a_kx^k), \tag{5}$$

where parameter k is always odd, while parameter l is natural number and for even l one can get even order of filter and for odd l odd order of filter.

The introduction of the zeros on polynomial transfer function will improve the amplitude characteristic and the cutoff slope but Chebyshev filter will loose minimax amplitude characteristic in the passband. In order to keep minimax characteristic in the passband it is necessary to find new coefficients of the polynomial $P_n(x)$.

Coefficients a_i of real polynomial $K_n(x)$ are obtained so that the characteristic function $K_n(x)$ has minimax characteristic in the passband. These coefficients can be calculated using an iterative procedure. More information about this calculation of coefficients of the polynomial $P_n(x)$ can be found in [5].

In table 1 coefficients of the polynomial $P_n(x)$ for two pairs of zeros on the unit circle in $\omega_z = 0.45\pi$, for different values of parameters k and l, are displayed.

When the polynomial $P_n(x)$ is determined, than the magnitude squared function can be written in the more convenient form

$$|H_{n,k,m}(e^{j\omega T})|^2 = \\ = \frac{1}{1+\varepsilon^2 x^{2l}(b_0+\cdots+b_{2k}x^{2k})\Big(\dfrac{x_z^2-1}{x^2-x_z^2}\Big)^{2m}}, \tag{6}$$

Table 1. Coefficients of polynomial $P_n(x)$, $n = 8$, $k = 0$, $m = 2$ and $\omega_z T = 0.45\pi$

l	0	2	4	6	8
k	8	6	4	2	0
a_0	3.825	-56.495	62.314	-17.508	1
a_2	-93.736	294.355	-155.343	18.508	—
a_4	399.622	-457.631	94.030	—	—
a_6	-573.442	220.771	—	—	—
a_8	264.732	—	—	—	—

where coefficients b_i can be obtained using the next simple relation

$$b_{2i} = \sum_{j=0}^{i} a_{2j} a_{2i-2j} \quad \text{for} \quad i = 0, ..., k. \tag{7}$$

After some rewriting the magnitude squared function can be written as

$$|H_{n,k,m}(e^{j\omega T})|^2 = \frac{(x^2 - x_z^2)^{2m}}{(x^2 - x_z^2)^{2m} + c_{2n}x^{2n} + \cdots + c_{2l}x^{2l}}, \tag{8}$$

where coefficients c_i can be obtained from the following relation

$$c_{2i} = (x_z^2 - 1)^{2m} \varepsilon^2 b_{2(i-1)}, \quad \text{for} \quad i = l, ..., n. \tag{9}$$

Substituting $x^2 = (zl)^2/(-4\alpha z)$ in (8) we obtain the next function

$$G(z) = \left(\frac{(z-1)^2}{-4\alpha z} - x_z^2\right)^{2m} \left[\left(\frac{(z-1)^2}{-4\alpha z} - x_z^2\right)^{2m} + \right.$$
$$\left. + c_{2n}\left(\frac{(z-1)^2}{-4\alpha z}\right)^n + \cdots + c_{2l}\left(\frac{(z-1)^2}{-4\alpha z}\right)^l\right]^{-1}, \tag{10}$$

which is equal to $|H_{n,k,m}(e^{j\omega T})|^2$ when evaluated along the unit circle and $\alpha = \sin^2(\omega T/2)$.

After some calculations in equation (10) the function $G(z)$ obtains form

$$G(z) = \frac{z^{n-2m}[z^2 + 2(2\beta - 1)z + 1]^{2m}}{z^{n-2m}[z^2 + 2(2\beta - 1)z + 1]^{2m} + F(z)}, \tag{11}$$

where parameter β is $\beta = \sin^2(\omega_z T/2)$ and

$$F(z) = f_0 z^{2n} + \cdots + f_n z^n + \cdots + f_0. \tag{12}$$

The function $F(z)$ is mirror-image polynomial because a sum of a mirror-image polynomial of order s and a mirror-image polynomial of order t multiplied by a mirror-image polynomial of order st is also a mirror-image polynomial of order s.

Coefficients f_i of the mirror-image polynomial $F(z)$ are given by

$$f_{2n-1} = (-4\alpha)^{2m} \sum_{j=0}^{2n-1} \frac{(-1)^j c_{2(i+j-n)} \binom{2(i+j-n)}{j}}{(-4\alpha)^{i+j-n}}, \tag{13}$$
$$\text{for} \quad i = n, n+1, ..., 2n.$$

The first part in the nominator of the transfer function (11) is also a mirror-image polynomial with multiple zeros on the unit circle and can be written in the extended form

$$[z^2 + 2(2\beta - 1)z + 1]^{2m} =$$
$$= z^{4m} + \cdots + e_{2m} z^{2m} + \cdots + 1, \tag{14}$$

where coefficients e_i can be obtained from the next relation

$$e_i = \sum_{j=0}^{i} \binom{2m}{j} \binom{4m - 2j}{i - j} (-1)^j (4\beta)^j, \tag{15}$$
$$\text{for} \quad i = 0, ..., 2m.$$

Collecting the same order coefficients of the mirror-image polynomials (12) and (14) we finally have

$$G(z) = \frac{z^{n-2m}[z^2 + (4\beta - 2)z + 1]^{2m}}{g_0 + \cdots + g_n z^n + \cdots + g_1 z^{2n-1} + g_0 z^{2n}}, \tag{16}$$

where coefficients g_i are given by

$$g_i = \begin{cases} f_i & \text{for} \quad i = 0, ..., n - 2k - 1 \\ f_i + e_{i-n+2k} & \text{for} \quad i = n - 2k, ..., n. \end{cases} \tag{17}$$

Equating the denominator of (16) with zero, the roots occur in reciprocal pairs. Now, it is easy to find poles because the poles of the transitional filters are merely roots inside the unit circle.

III. Results of Approximation

Normalized frequency response of the new filter of eighth order and one pair of zeros at $\omega_z T = 0.45\pi$ is displayed in Figure 1. Normalized cutoff frequency is $\omega_c T = 0.3\pi$, and maximal passband attenuation is $A_{\max} = 1$ dB. Passband attenuation ripples are equalized. As parameter l is lower, the selectivity of filter is improved. Minimal stopband attenuation significantly differs and depends from the value of the parameter l, so for $l = 8$ minimal stopband attenuation is

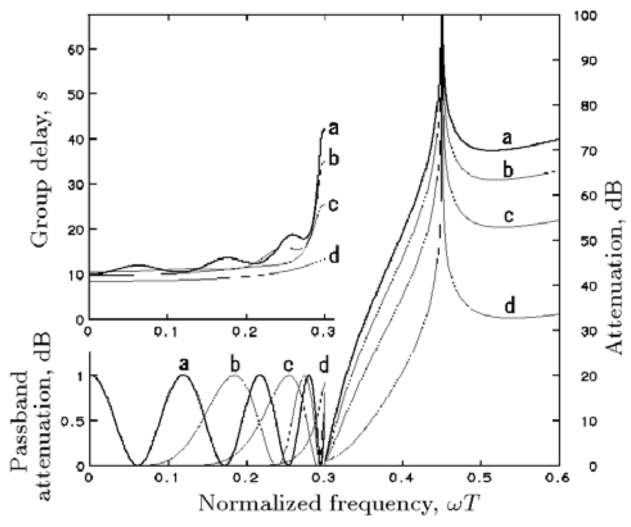

Fig. 1. Characteristics of transitional filters for different values of parameters l; $m = 1$, $\omega_c T = 0.3\pi$, $A_{\max} = 1$ dB; a) $l = 0$, b) $l = 4$, c) $l = 6$ and d) $l = 8$

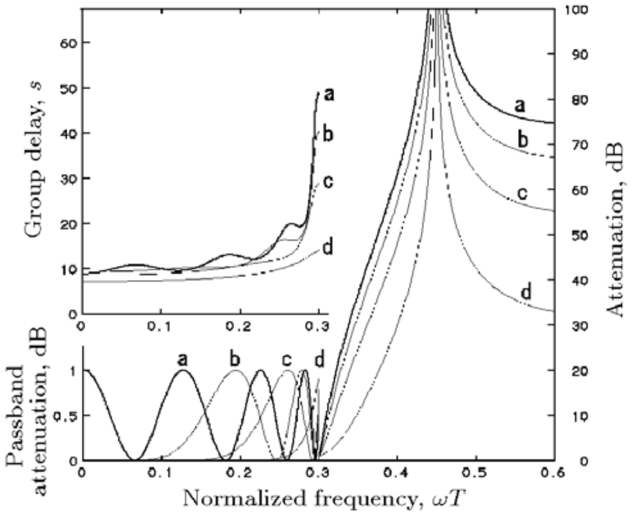

Fig. 2. Characteristics of transitional filters for different values of parameters l; $m = 2$, $\omega_c T = 0.3\pi$, $A_{\max} = 1$ dB; a) $l = 0$, b) $l = 4$, c) $l = 6$ and d) $l = 8$

near 32 dB, while for $l = 0$ it is slightly less than 70 dB. Minimums of stopband attenuation are all at the same value of normalized frequency $\omega_p T$. Group delay characteristics has been also displayed and it is obviously that the characteristic becomes flatter increasing parameter l. For $l = 8$ the group delay is in the range between 9 s and 11 s and for $l = 0$ it is between 10 s and 42 s.

Influence of the multiplicity order of zeroes m at the unit circle to the frequency characteristic of filter is shown in Figure 2. The frequency characteristic of the filter with two pairs of zeros at the unit circle in $\omega_z = 0.45\pi$ is displayed. In order to compare frequency characteristics, normalized cutoff frequency and maximal passband attenuation are the same as in the previous example. Increasing multiplicity order m, selectivity also increases and minimal stopband attenuation is bigger. For $l = 0$ minimal stopband attenuation is more than 75 dB, while for $l = 8$ it is about 34 dB. Group delay characteristic has more deviation comparing with previous example. For $l = 8$ group delay is now in the range from 8 s until 12 s and for $l = 0$ it is between 9 s and 48 s.

Differentiating (1) in ωT, it is obviously that the slope of the amplitude characteristics in the frequency $\omega T = \omega_c T$ is given by

$$S_{n,k} = \frac{-\varepsilon^2 \sum\limits_{i=0}^{k} a_i \left\{ \sum\limits_{i=0}^{k} (i+l)a_i + \dfrac{2m}{x_z^2 - 1} \sum\limits_{i=0}^{k} a_i \right\}}{2\tan\left(\dfrac{\omega_c T}{2}\right)\sqrt{(1+\varepsilon^2)^3}} \qquad (18)$$

Sum of coefficients of the polynomial $P_n(x)$ depends from multiplicity order of introduced zeroes and has value $(1)^m$. It means that for even m it is equal to 1, and for odd m it is equal to 1.

The cutoff slope of the amplitude characteristic depends from the width of the passband and it is smaller if the passband is wider. If the normalized value of the passband is π the cutoff slope is equal to zero. From above reasons this approximation is more suitable for design of narrow band low-pass digital filters. Introducing simple or multiple zero pairs on frequency $\omega_z T > \omega_c T$ it is possible to obtain satisfactory value for the slope of the amplitude characteristics on the cutoff frequency even for the low pass digital filters with the wider passband.

We can conclude from equation (18) that if the cutoff frequency $\omega_c T$ and the order of filter n are known, we can choose the cutoff slope changing four different parameters. These parameters are:

- Parameter l (i.e. parameter k),
- Zero of the transfer function $\omega_z T$,
- Multiplicity order of zero pairs m and
- Maximal passband attenuation $A_{\max}$.

The cutoff slope will be increased if parameter l is lower, $\omega_z T$ is closer to the cutoff frequency $\omega_c T$, multiplicity of the zero pairs m is bigger and if the maximal passband attenuation is bigger.

This conclusion can be confirmed from Figure 3. Dependency of the slope of the amplitude characteristics from the passband value $\omega_c T$ for transitional filters with $m = 2$ zero pairs on the unit circle at $\omega_z T = \omega_c T + 0.1\pi$ has been shown in this Figure. The order of filters is $n = 8$. The cutoff slope decreases if the passband is wider. It confirmed our conclusion that this type of filters is more suitable for design of narrow band low pass digital filters.

In order to check influence of zero pairs on the unit circle on the cutoff slope, in Figure 4 is displayed cutoff slope dependency from value of the $\omega_z T$. We can see that if the $\omega_z T$ is closer to the boundary of the passband, the cutoff slope will be increased. The cutoff slope will increase also if the parameter l becomes lower. The cutoff slopes of filters for $l = 0$ and $l = 2$ are very similar and it is a consequence of the similar frequency characteristics.

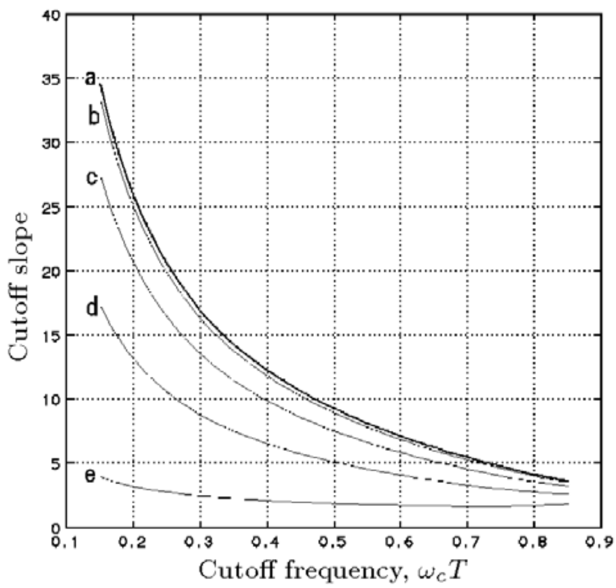

Fig. 3. Cutoff slope characteristics for different values of cutoff frequency $\omega_c T$; $A_{\max} = 1$ dB; a) $l = 0$, b) $l = 2$, c) $l = 4$, d) $l = 6$, e) $l = 8$

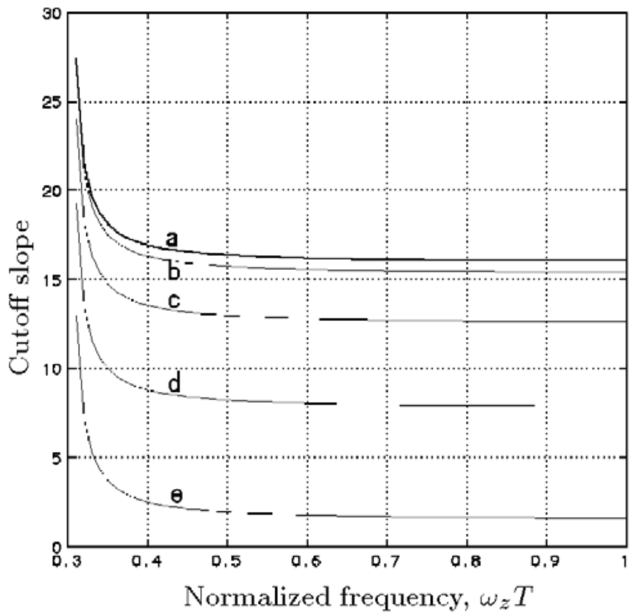

Fig. 4. Cutoff slope characteristics for different values of of parameter $\omega_c T$; $A_{\max} = 1$ dB; a) $l = 0$, b) $l = 2$, c) $l = 4$, d) $l = 6$, e) $l = 8$

IV. Conclusion

In this paper a procedure for design of transitional Butterworth-Chebyshev digital filters with a single or multiple zero-pairs on the unit circle and mini-max amplitude characteristic in the passband has been described. These filters can not be obtained using transformations from analogous domain. Proposed procedure is suitable for design of narrow band low pass digital filters.

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