Invariant Subspaces in Third-Order Digital Filters with Two's Complement Overflow

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Abstract – **In this paper we propose a general approach for locating the invariant subspaces of some classes of third order digital filters with two's complement overflow. The proposed approach is based on the analysis of the projection of the points of the trajectories on adequately chosen directions in the filter phase space.**

Keywords – **Digital filter, two's complement overflow, attractor, invariant space, phase space, eigenvalue, and eigenvector**

I. Introduction

Nonlinearities in digital filters and their influences on their dynamics have attracted considerable attention in the last fifteen years. The very first works in this field [1,2], show that even the simple systems like second order digital filters can exhibit complex behavior due their two's complement overflow nonlinearity. This phenomenon is manifested by existence of various types of trajectories in the digital filter's phase space which depend both, on the filter parameters, and by their initial starting points.

In the farther papers, special attention was attended on the third and the higher order digital filters operating out of their stability region [3,4]. In these papers it was noticed that the trajectories of the digital filters usually non-uniformly visit various parts of the phase space, and in some cases they visit regularly only some invariant-subspaces of the phase space.

In this paper we concentrate on the third order digital filters with two's complement overflow nonlinearity, operating outside of their linear stability region. For this type of filters we develop a new, geometry oriented approach for localization of the invariant subspaces visited by their trajectories. The proposed approach is based on the analyses of the projection of the trajectories of the digital filter on suitably chosen directions in its phase space, which depend on the filter parameters.

The material in this paper is organized as follows. The piece wise linear model of the third order digital filter is given in Section II. In Section III, we develop expression for the projection of the k-th iteration of a map on an arbitrary vector in the phase space, and determine the criteria when the trajectories of the digital filters visit finite number of planes into the phase space. In Section IV we consider situation when the Jacobijan of the map has at least one eigenvalue inside and one outside of the unit circle, and determine the conditions when the attractors of the map are localized in some compact parts (subspaces) of the phase spaces. In Section V we illustrate and discuss our results and at last, in Section VI we give some conclusions.

II. Piece-Wise Linear Model

The behavior of zero input, third order digital filter with 2's complement overflow can be described by the following system of difference equations

$$\begin{aligned} x_1(k+1) &= x-2(k) \\ x_2(k+1) &= x_3(k) \\ x_3(k+1) &= f(a_1x_1(k)+a_2x_2(k)+a_3x-3(k)) \end{aligned} \quad (1)$$

where: $f(\cdot)$ is an almost odd function defined by:

$$f(x) = x-2s \quad \text{for} \quad -l+2s \geq x < l+2s; \quad (2)$$

a_1, a_2, a_3 are filter parameters; and $x_1(k)$, $x_2(k)$ and $x_3(k)$ are the filter internal states.

The behavior of the filter, can be also described by the piece-wise linear map $\mathbf{F}(x(k)) : \mathbf{I}^3 \to \mathbf{I}^3$,

$$\mathbf{x}(k+1) = \mathbf{F}(f(\mathbf{a}^T\mathbf{x}(k)) = \mathbf{A}\mathbf{x}(k) + \mathbf{b}s \text{ for } \mathbf{x}(k) \in \mathbf{I}_s^3 \quad (3)$$

where:

$$\mathbf{x}(k) = \begin{vmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{vmatrix}; \quad \mathbf{a} = \begin{vmatrix} a_1 \\ a_2 \\ a_3 \end{vmatrix}; \quad ||\mathbf{a}|| = |a_1|+|a_2|+|a_3|;$$

$$\mathbf{A} = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_1 & a_2 & a_3 \end{vmatrix}; \quad \mathbf{b} = \begin{vmatrix} 0 \\ 0 \\ 2 \end{vmatrix};$$

$$\mathbf{I}^3 = \{\mathbf{x} = [x_1, x_2, x_3]^T : -1 \leq x_i < 1,\ i = 1,2,3\};$$

$$\mathbf{I}_s^3 = \{\mathbf{x} : -2s-1 \leq \mathbf{a}^T\mathbf{x} < -2s+1,\ \mathbf{x} \in \mathbf{I}^3, s \in Z^*\};$$

$$Z^* = \{s_{\min}, s_{\min}+1, ..., 0, 1, ..., s_{\max}\};\ s_{\max} = -s_{\min};$$

and

$$s_{\max} = \begin{cases} \left[\dfrac{||\mathbf{a}||+1}{2}\right]; & \text{for } ||\mathbf{a}|| \neq 2p+1 \\ \left[\dfrac{||\mathbf{a}||+1}{2}\right]-1; & \text{for } ||\mathbf{a}|| = 2p+1 \end{cases}; \ p = 0, 1, ...$$

This means that $\mathbf{I}^3$ is divided into $2s_{\max}+1$ subspaces $\mathbf{I}_s^3$, $s \in Z^*$, separated by $2s_{\max}$ parallel planes π,

$$\pi = \{\mathbf{x} : \mathbf{a}^T\mathbf{x} = 2s-1, s \in \{Z^* \backslash s_{\max}\}\}, \quad (4)$$

(as illustrated in Fig. l).

By using k recursive iterations in (3), one can obtain the expression for the k-th iteration of the map

$$\mathbf{x}(k) = \mathbf{A}^k\mathbf{x}(0) + \sum_{j=0}^{k-1} \mathbf{A}^{k-1-j}\mathbf{b}s_j, \quad (5)$$

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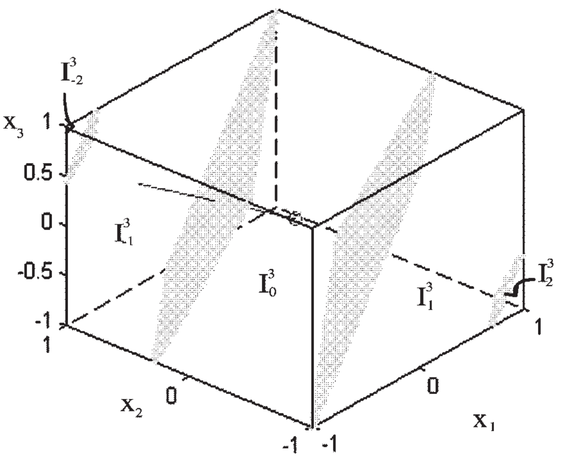

Fig. 1. Subspaces of the filter phase space I^3 for the filter with parameters $\mathbf{a} = (a_1, a_2, a_3) = (-1.17, 1.57, 0.6)$ (eigenvalues of $\mathbf{A}$ are $\lambda_1 = 1$, $\lambda_2 = -1.3$, $\lambda_3 = 0.9$)

in which s_j $(j = 0, l, ..., k - l)$ is an integer that corresponds to the index of the subspace $\mathbf{I}^3_{s_j}$ visited by the j-th iteration. Hence, each trajectory of the map (3), starting from any initial point $\mathbf{x}(0) = \mathbf{x}_0$, $\mathbf{x}_0 \in \mathbf{I}^3$, generates infinite symbolic sequence $s_0 s_1 ... s_j ...$ composed of the indexes of the visited subspaces.

III. Invariant Planes

The characteristics of each trajectory, given by Eq. (5), depend on both, the initial point $\mathbf{x}(0)$ and the properties of the map which are determined by the Jacobijan matrix of the map $\mathbf{F}(\mathbf{x})$, $D(\mathbf{F}(\mathbf{x})) = \mathbf{A}$.

We analyze the behavior of the map (3) under assumption that the matrix $\mathbf{A}$ has full rank, $\text{rank}(\mathbf{A}) = 3$, and that it has 3 distinct eigenvalues λ_1, λ_2 and λ_3 with corresponding eigenvectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$. Without any constrains on the eigenvalues of $\mathbf{A}$, the projection of the k-th (k-arbitrary non-negative integer) iteration of the map on a direction defined by an arbitrary unit vector $\mathbf{v}$ in $\mathbf{R}^3$, $\mathbf{v} \in \mathbf{R}^3$, is

$$\mathbf{v}^T\mathbf{x}(k) = \mathbf{v}^T\mathbf{A}^k\mathbf{x}_0 + \mathbf{v}^T\sum_{j=0}^{k-1}\mathbf{A}^{k-1-j}\mathbf{b}s_j, \qquad (6)$$

$$\mathbf{v}^T\mathbf{x}(k) = ((\mathbf{A}^k)^T\mathbf{v})^T\mathbf{x}_0 + \sum_{j=0}^{k-1}((\mathbf{A}^{k-1-j})^T\mathbf{v})^T\mathbf{b}s_j. \qquad (7)$$

If $\mathbf{v}$, is an unit eigenvector of $\mathbf{A}^T$ that corresponds to the eigenvalue λ_j, ($\mathbf{A}_i^T\mathbf{v}_i = \lambda_i\mathbf{v}_i$; $i = l, 2, 3$), then the last equation becomes

$$\mathbf{v}_i^T\mathbf{x}(k) = \lambda_i^k\mathbf{v}_i^T\mathbf{x}_0 + \mathbf{v}_i^T\mathbf{b}\sum_{j=0}^{k-1}\lambda_i^{k-1-j}s_j. \qquad (8)$$

By denoting $\mathbf{v}_i^T\mathbf{x}_0 = \alpha_i(\mathbf{x}_0)$ and $\mathbf{v}_i^T\mathbf{b} = \beta_i$, the last equation can be rewritten as

$$\mathbf{v}_i^T\mathbf{x}(k) = \alpha_i(\mathbf{x}_0)\lambda_i^k + \beta_i\sum_{j=0}^{k-1}\lambda_i^{k-1-j}s_j. \qquad (9)$$

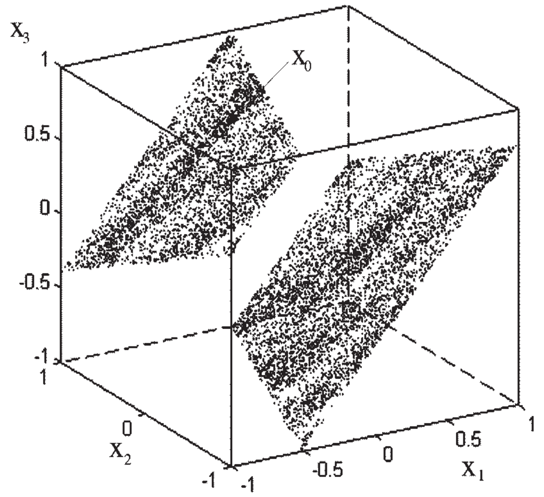

Fig. 2. Trajectory of the filter with parameters $\mathbf{a} = (a_1, a_2, a_3) = (-1.17, 1.57, 0.6)$ (eigenvalues of $\mathbf{A}$ are $\lambda_1 = 1$, $\lambda_2 = -l.3$, $\lambda_3 = 0.9$), starting from $\mathbf{x}_0 = [-0.34, 0.23, 0.7]^T$

This equation is crucial for our further analysis of the invariant subspaces of the map (3).

If $\lambda_1 = 1$, then the last equation becomes

$$\mathbf{v}_1^T\mathbf{x}(k) = \alpha_1(\mathbf{x}_0) + \beta_i\sum_{j=0}^{k-1}s_j. \qquad (10)$$

Since $\mathbf{x}(k) \in \mathbf{I}^3$ and $s_0 + s_1 + \cdots + s_k$ is an integer, it implies that $s_0 + s_1 + \cdots + s_k$ must be a finite integer for each k. Therefore, one can conclude that $\mathbf{x}(k)$ belongs to a finite set of parallel planes Ω_1 normal on $\mathbf{v}_1^T$ [5], defined by

$$\Omega_1(\mathbf{x}_0) = \{\mathbf{x} : \mathbf{v}_1^T\mathbf{x}(k) = \alpha_1(\mathbf{x}_0) + \beta_i\sum_{j=0}^{k-1}s_j,\ k = 0, 1, ...\}. \qquad (11)$$

Hence, for each initial point $\mathbf{x}_0$, we can determine the invariant set of points

$$\Theta_1(\mathbf{x}_0) = \{\Omega_1(\mathbf{x}_0) \cap \mathbf{I}^3\} \qquad (12)$$

that satisfy $\mathbf{F}^k(\Theta_1(\mathbf{x}_0)) \subseteq \Theta_1(\mathbf{x}_0)$; $k = 0, 1, 2, ...$ It means that each trajectory that starts from any point of this set of parallel planes remains on them. This property is illustrated in Fig. 2 in which we present the first 20 000 points of the trajectory of the filter (defined by the eigenvalues $\lambda_1 = 1$, $\lambda_2 = -1.3$ and $\lambda_3 = 0.9$ of the matrix $\mathbf{A}$), starting from $\mathbf{x}_0 = [-0.34, 0.23, 0.7]^T$.

Similar result could be obtained for $\lambda_1 = -1$. In this case Eq. (8) becomes

$$\mathbf{v}_1^T\mathbf{x}(k) = (-1)^k\alpha_1(\mathbf{x}_0) + \beta_1\sum_{j=0}^{k-1}(-1)^{k-1-j}s_j. \qquad (13)$$

Since $\mathbf{x}(k) \in \mathbf{I}^3$, it implies that $\mathbf{x}(k)$ belongs to a finite set of parallel planes $\Omega_2(x_0) \cup \Omega_3(x0)$ normal on $\mathbf{v}_1^T$, where

$$\Omega_2(\mathbf{x}_0) = \{\mathbf{x} : \mathbf{v}_1^T\mathbf{x} = \alpha_1(\mathbf{x}_0) + \beta_1 \sum_{j=0}^{k-1} (-1)^{k-1-j} s_j,\ k - \ odd\}; \quad (14)$$

$$\Omega_3(\mathbf{x}_0) = \{\mathbf{x} : \mathbf{v}_1^T\mathbf{x} = -\alpha_1(\mathbf{x}_0) + \beta_1 \sum_{j=0}^{k-1} (-1)^{k-1-j} s_j,\ k - \ even\}. \quad (15)$$

Hence, for each initial point $\mathbf{x}_0$, we can determine the invariant set of points

$$\Theta_2(\mathbf{x}_0) = \{(\Omega_2(\mathbf{x}_0) \cup \Omega_3(\mathbf{x}_0)) \cap \mathbf{I}^3\} \quad (16)$$

that satisfy $\mathbf{F}^k(\Theta_2(\mathbf{x}_0) \subseteq \Theta_2(\mathbf{x}_0);\ k = 0, 1, 2, \ldots$

If $\mathbf{A}$, has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 1$, then the invariant space of the map (3) becomes a set of parallel lines collinear with the vector $(\mathbf{v}_1^T \times \mathbf{v}_2^T)$, that intersects $\mathbf{I}^3$

$$\Lambda(\mathbf{x}_0) = \{\Omega_1(\mathbf{x}_0) \cap (\Omega_2(\mathbf{x}_0) \cup \Omega_3(\mathbf{x}_0))\} \cap \mathbf{I}^3. \quad (17)$$

IV. Invariant Subspaces

It is obvious that the behavior of the trajectories of the map (3), depend on the eigenvalues of the Jacobijan matrix $\mathbf{A}$. If $\mathbf{A}$ has at least one eigenvalue out of the unit circle then each trajectory of the map is either periodic or chaotic. If the trajectories are chaotic then they are all attracted by attractors which are very strangely positioned in the phase space.

In this section we analyze the subspaces of the phase space in which are placed all attractors in cases when the Jacobijan matrix $\mathbf{A}$ of the map has at least one eigenvalue in the interior of the unit circle, say $|\lambda_2| < 1$, and one eigenvalue out of the unit circle, say $|\lambda_3| > 1$.

In this case, if $\mathbf{v}_2$ is the eigenvector of $\mathbf{A}^T$ that corresponds to λ_2, the Eq. (8) becomes

$$\mathbf{v}_2^T\mathbf{x}(k) = \lambda_2^k \alpha_2(\mathbf{x}_0) + \beta_2 \sum_{j=0}^{k-1} \lambda_2^{k-1-j} s_j. \quad (18)$$

For sufficiently large k, Eq. (l8) could be further reduced to

$$\mathbf{v}_2^T\mathbf{x}(k) = \beta_2 \sum_{j=0}^{k-1} \lambda_2^{k-1-j} s_j, \quad (19)$$

since $|\lambda_2| < 1$ and $\lim_{k\to\infty} \alpha_2(\mathbf{x}_0)\lambda_2^k = 0$.

The last equation can be rearranged in the following form

$$\frac{\mathbf{v}_2^T\mathbf{x}(k)}{\beta_2} - s_{k-1} = \lambda_2(s_{k-2} + s_{k-3}\lambda_2 + \cdots + s_0\lambda_2^{k-2}). \quad (20)$$

By using the fact that $-s_{\max} \le s_j \le s_{\max}$, we obtain

$$\left|\frac{\mathbf{v}_2^T\mathbf{x}(k)}{\beta_2} - s_{k-1}\right| < |\lambda_2|\, |s_{\max}| \frac{1 - |\lambda_2|^{k-1}}{1 - |\lambda_2|} < |s_{\max}| \frac{|\lambda_2|}{1 - |\lambda_2|} = R(\lambda_2), \quad (21)$$

from which we conclude that the projection of the k-th iterate of the map (3) on the unit vector $\mathbf{v}_2^T$ belongs to the interval

$$\mathbf{X}_{s_{k-1}} = \Big(\beta_2 s_{k-1} - |\beta_2 R(\lambda_2)|,\ \beta_2 s_{k-1} + |\beta_2 R(\lambda_2)|\Big), \quad (22)$$

($s_{k-1} \in [-s_{\max}, s_{\max}]$), determined by the index of the subspace visited in the $(k - l)$-st iteration of the map. Since the number of subspaces $\mathbf{I}_{s_j}^3$ is $2s_{\max} + 1$, the number of intervals (22) is also $2s_{\max} + 1$. If $R(\lambda_2) < 0.5$, then there will be some gaps between those intervals. Therefore, all the attractors of the map (3) will belong to the union of $2s_{\max} + 1$ disjointed subspaces of $\mathbf{I}^3$, ($\Psi_s \subset \mathbf{I}^3$), defined by

$$\Omega_4 = \bigcup_{s=-s_{\max}}^{s_{\max}} \Psi_s, \quad (23)$$

where $\Psi_s(\mathbf{x}) = \{\mathbf{x} : d_l(s) < \mathbf{v}_2^T\mathbf{x} < d_h(s);\ \mathbf{x} \in \mathbf{I}^3\}$; $s_{k-1} \in [-s_{\max}, s_{\max}]$, $d_l(s) = \beta_2 s - |\beta_2 R(\lambda_2)|$ and $d_h = \beta_2 s + |\beta_2 R(\lambda_2)|$.

It means that each point of any trajectory of the map, after certain iteration, will belong to some parts of the phase space that are bounded by the planes $\gamma_l = \{\mathbf{x} : \mathbf{v}_2^T\mathbf{x} = d_l(s)\}$ and $\gamma_h = \{\mathbf{x} : \mathbf{v}_2^T\mathbf{x} = d_h(s)\}$ normal on $\mathbf{v}_2^T$.

In this case, the set Ω_4 doesn't depend on the initial condition $\mathbf{x}_0$ and it is definitely the invariant set of the phase space since $\mathbf{F}^k(\Omega_4) \subseteq \Omega_4$ for any k.

V. Simulations

In this section we illustrate our analytical results by functions of the histograms of the projections of the trajectories of two suitably chosen third order digital filters that belong to the observed classes of filters.

In Fig. 3, we present the histograms of the projections of the trajectory of the filter with parameters $\mathbf{a} = (-1.77, 1.57, 0.6)$ (starting from $\mathbf{x}_0 = [-0.34, 0.23, 0.7]^T$) on the eigenvectors of the transpose of the Jacobijan of the map that describes the behavior of the filter. The Jacobijan matrix has the following eigenvalues: $\lambda_1 = l$, $\lambda_2 = 1.3$, and $\lambda_3 = 0.9$.

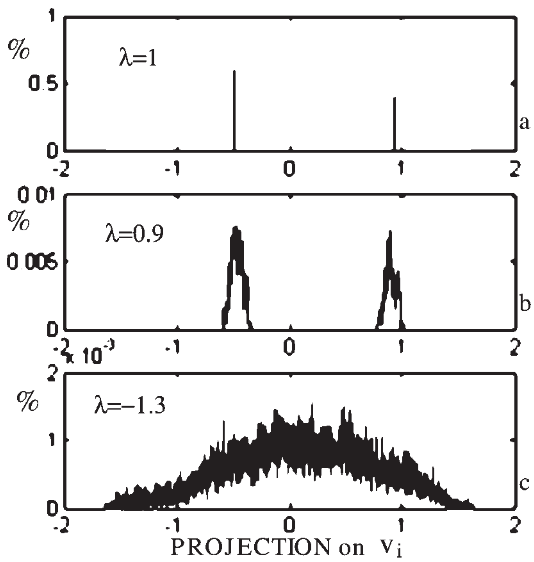

Fig. 3. Histograms of the projections of the first 20 000 points of the trajectory of the filter on the eigenvectors of $\mathbf{A}^T$, binned into 1001 containers: a) projection on $\mathbf{v}_1$, b) on $\mathbf{v}_2$, c) on $\mathbf{v}_3$

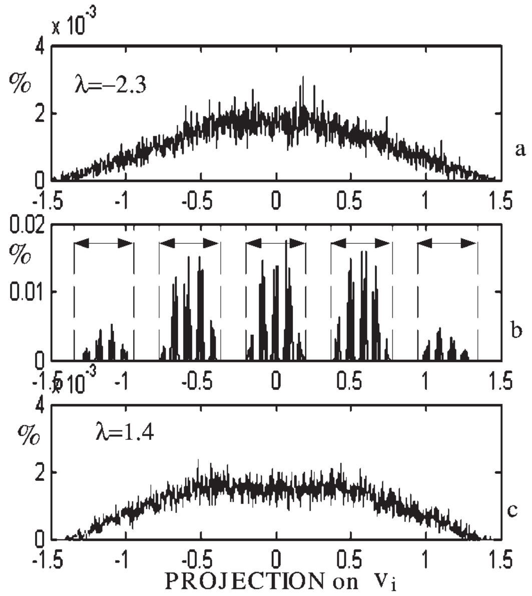

Fig. 4. Histograms of the projections of the first 20 000 points of the trajectory of the filter with eigenvalues of $\mathbf{A}$: $\lambda_1 = -2.3$, $\lambda_2 = 0.15$, $\lambda_3 = 1.4$ on the eigenvectors of $\mathbf{A}^T$, binned into 1001 containers: a) projection on $\mathbf{v}_1$, b) on $\mathbf{v}_2$, c) on $\mathbf{v}_3$

The histogram of the projection of the trajectory on the eigenvector $\mathbf{v}_1$ of $\mathbf{A}^T$ that correspond to $\lambda_1 = 1$ is discrete with two clearly defined picks that correspond to projection of the planes (Fig. 2) visited by the trajectory.

The histogram of the projection on the eigenvector $\mathbf{v}_2$ of $\mathbf{A}^T$ that corresponds to $\lambda_2 = 0.9$, $(|\lambda_2| < 1)$ is localized in two narrow isolated intervals, while the histogram of the projection of the trajectory on the eigenvector $\mathbf{v}_2$ of $\mathbf{A}^T$ that corresponds to $\lambda_3 = -1.3$ is non-uniformly speeded all over the projection domain.

Although this filter does not satisfy the constrain $R(\lambda_2) < 0.5$, the projection of its trajectory on $\mathbf{v}_2$ is concentrated on a narrow region due to the nature of the admissible symbolic sequences that correspond to the points of the phase space. Numerous simulations have shown us that the width of the isolated nonzero intervals of the histogram will start to spread (and eventually it will begin to fractalize) by decreasing the module of the eigenvalue $|\lambda_2|$ toward zero. When the condition $R(\lambda_2) < 0.5$ is reached, the trajectory of the filter enters the region Ω_4 and remains there.

In Fig. 4 we present the histograms of the projections of the trajectory of a filter described by a map which Jacobijan have the following eigenvalues: $\lambda_1 = -2.3$, $\lambda_2 = 0.15$ and $\lambda_3 = 1.4$.

In this case the condition $R(\lambda_2) < 0.5$ is satisfied. Therefore the histogram of the projection of the trajectory on $\mathbf{v}_2$ is located in five intervals $\mathbf{X}_s$, $s = -2, -l, 0, l, 2$, with gaps among them as illustrated in Fig. 4b. As $R(\lambda_2) \to 0$, the width of the intervals with nonzero histogram functions starts to decrease (and fractalize) in favor of the gaps between them.

The nature of the histograms of the projections of trajectories on eigenvectors that correspond to eigenvalues that are out of the unit circle is out of the scope of this paper.

VI. Conclusions

In this paper we have analytically located the invariant subspaces of some classes of third order digital filters with two's complement overflow by using the eigenvalues of the Jacobijan which describes the behavior of the filter. In cases of real eigenvalues that belong to the unit circle, the invariant subspaces are sets of parallel planes determined by the starting point of the trajectories of the filter. In case of eigenvalues with sufficiently small modules, the invariant subspaces are set of slices of the phase (each bounded by two parallel planes) which are independent of the starting point of the trajectories.

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