Inverse Hausdorf LC Filters

Peter S. Apostolov[1]

Abstract – **This paper presents a method for synthesis of inverse filters with Hausdorf-type of transmission characteristics. The frequency characteristics of the filters are determined and a comparison with inverse Chebyshev filters is done.**

Keywords – **approximation, Hausdorf polynomial, synthesis, LC filter, frequency characteristic**

I. Introduction

Hausdorf filters are implemented via an approximation of the "shifted" Delta-function with a Hausdorf polynomial of the following type:

$$P_n(\omega) = \varepsilon T_n\left(\frac{2\omega}{2-\alpha\varepsilon}\right), \tag{1}$$

where T_n is a Chebyshev polynomial, ε is the best approximation of a "shifted" delta function, α is parameter, defining the hold bandwidth of the filter, ω is the angle frequency. Low-pass non-inverse filters synthesized using this type of approximation have parameters identical to the Chebyshev filters' parameters. The pass bandwidth of these filters is narrowed by a coefficient equal to one half of the Hausdorf distance $\alpha\varepsilon$ [3].

Given the filter order n and passband ripple DA [dB], the Hausdorf space ε and the argument α could be found via the following equations [2]:

$$\varepsilon = \sqrt{10^{0.1DA}-1}, \tag{2}$$

$$\alpha\varepsilon = 2\frac{\cosh\left[\frac{1}{n}\text{arccosh}\left(\frac{1}{\varepsilon}\right)\right]-1}{\cosh\left[\frac{1}{n}\text{arccosh}\left(\frac{1}{\varepsilon}\right)\right]+1}. \tag{3}$$

The inverse low-pass Hausdorf filters are of interest from the syntheses point of view. They have a maximally flat passband and an even ripple stopband.

II. Synthesis Implementation

The synthesis has the following prerequisites: filter order n, cut-off frequency f_c and its transmission function attenuation $k = \sqrt{10^{0.1DA}-1}$, stopband frequency f_s and the Hausdorf space $\alpha\varepsilon$.

The Hausdorf space $\alpha\varepsilon$ can be derived by calculating the transmission function of an inverse Chebyshev filter of the same order for the stopband frequency:

$$DS_{ch} = \log\left[k^2\cosh^2\left(n\text{arccosh}\frac{f_s}{f_c}\right)\right]. \tag{4}$$

[1]Peter S. Apostolov is with the Institute for Special Technical Equipment-MI, E-mail: p_apostolov@abv.bg

Next, the ε of its low-pass filter prototype is determined as:

$$\varepsilon = \frac{1}{\sqrt{10^{0.1DS}-1}}$$

and from equation (3) $\alpha\varepsilon$ is derived.

The square of the transmission function module is derived from the characteristic function, which in this case is the Hausdorf polynomial:

$$|A|^2 = \frac{k^2T_n^2\left(\frac{2-\alpha\varepsilon}{2\omega}\right)}{1+k^2T_n^2\left(\frac{2-\alpha\varepsilon}{2\omega}\right)}. \tag{5}$$

The two functions can be described as relations between the three polynomials $e(s)$, $p(s)$ and $q(s)$ of the complex frequency $s = j\omega$. The polynomial $e(s)$ is a strict Hurvitz polynomial and its roots are the fundamental frequencies of the filter, the roots of $p(s)$ are extreme frequencies for which the transmission function has infinite attenuation. The answer to the synthesis problem is usually found by determining two of the polynomials, the third polynomial is the result of the Feldtkeller equation:

$$e(s)e(-s) = p(s)p(-s) + q(s)q(-s). \tag{6}$$

The roots of $e(s)$ and $p(s)$ are derived as follows:

$$S_p = \frac{\left(1-\frac{\alpha\varepsilon}{2}\right)}{(Q_1+jQ_2)}, \tag{7}$$

where

$$\begin{aligned} Q_1 &= -\sin\left(\frac{2i-1}{n}\frac{\pi}{2}\right)\sinh\left[\frac{1}{n}\text{arcsinh}\left(\frac{1}{k}\right)\right]; \\ Q_2 &= \cos\left(\frac{2i-1}{n}\frac{\pi}{2}\right)\cosh\left[\frac{1}{n}\text{arcsinh}\left(\frac{1}{k}\right)\right]; \\ S_n &= \frac{j\left(1-\frac{\alpha\varepsilon}{2}\right)}{\cos\left(\frac{2i-1}{n}\frac{\pi}{2}\right)}, \quad (i = 1 \div n) \end{aligned} \tag{8}$$

The value of the transmission function, when the cut-off frequency and the stopband frequencies are known, is:

$$DS_h = -20\log\sqrt{\frac{k^2\cosh^2\left[n\,\text{arccosh}\frac{f_c(1-\alpha\varepsilon/2)}{f_s}\right]}{1+k^2\cosh^2\left[n\,\text{arccosh}\frac{f_c(1-\alpha\varepsilon/2)}{f_s}\right]}}. \tag{9}$$

The value of the elements is calculated using the methodology described in [4] by transforming the variable s into a new variable z:

$$z^2 = 1 + \frac{\omega_c^2}{s^2} \tag{10}$$

Based on this methodology, two different software programs for filter calculation are offered in [3], called APPROX and LC. When entering the input data, the frequencies resulting from (8) need to be used.

III. Frequency Characteristics of an Inverse Hausdorf Filter

A. *Magnitude (amplitude) response*

Using the synthesis method described above, a low-pass Hausdorf filter of third order ($n = 3$) has been calculated assuming that the cut-off frequency is 10 KHz, the stopband frequency is 15 KHz, 0.3 dB attenuation at f_c, input and output resistance of 1 Ω. The electrical circuit is shown in Fig. 1, and a computer simulation of the magnitude response of the filter is shown in Fig. 2. The magnitude response of an inverse Chebyshev Filter with the same input specification is shown in Fig. 3. The comparison of the two shows that the Hausdorf filter has a steeper slop in $(f_c \div f_\infty)$ interval. In this case the attenuation of the stopband frequency is greater than 4.2 dB. The maximum in the passband is greater than 1.85 dB. This is due to the fact that the extreme frequencies of the inverse Hausdorf Filter are $(1-\alpha\varepsilon/2)$ times lower than these of the inverse Chebyshev filter, as seen in equation (8).

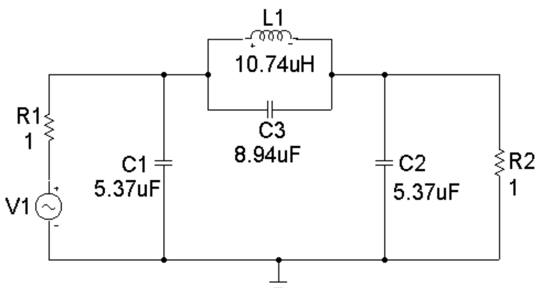

Fig. 1.

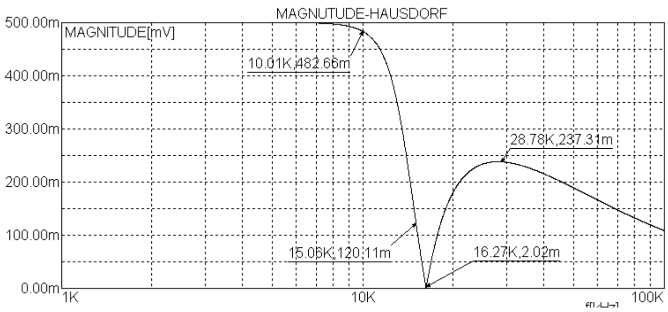

Fig. 2.

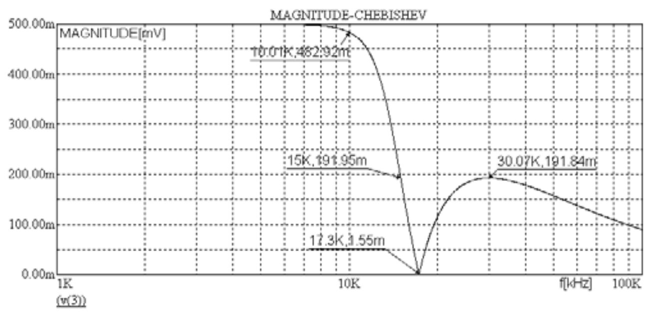

Fig. 3.

B. *Phase-frequency response*

Polynomial $e(s)$ is determined by solving the equation (7). Shown as a rational function, after substituting $s = j\omega$, it is broken into real e_R and imaginary e_I polynomials. The same is done for the polynomial $p(s)$, derived in (7). The Phase-frequency Response is:

$$\varphi(\omega) = \arctan\frac{p_I}{p_R} - \arctan\frac{e_I}{e_R}. \tag{11}$$

The phase characteristics of Hausdorf and Chebyshev filters are shown on Figs. 4 and 5. They point out that the Hausdorf filters have worse linearity – it is 5.36% for the marked frequencies.

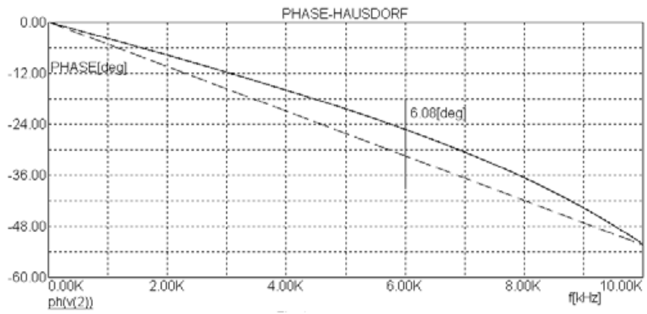

Fig. 4.

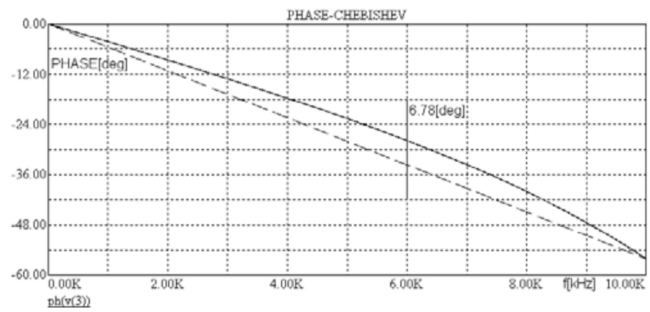

Fig. 5.

C. *Group delay time (GDT)*

GDT is calculated from:

$$t_{gr} = \mathrm{Re}\Big[\frac{1}{e(s)}\frac{de(s)}{ds} - \frac{1}{p(s)}\frac{dp(s)}{ds}\Big]. \tag{12}$$

Figs. 6 and 7 show the GDT graphs of the filters described in the previous paragraph. It follows from the graph that the Hausdorf filter has on overall lower values for t_{gr}, meaning that it has lower reactivity. Comparing the two graphs, it is calculated that the Hausdorf filter graph is 13.6% more non linear.

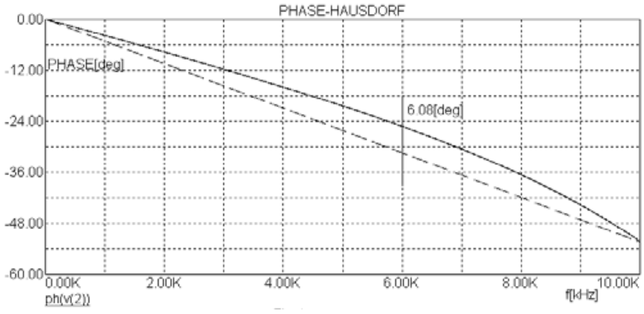

Fig. 6.

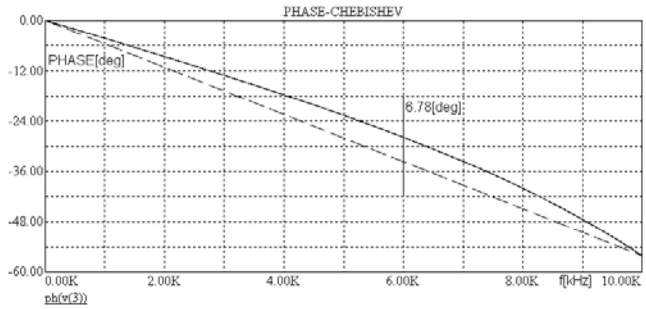

Fig. 7.

IV. Conclusion

The specific characteristics for the Hausdorf filters show that they can be used in circuits requiring high attenuation for frequencies close to the cut-off frequencies. Their lower reactivity is a prerequisite for their use in cases when a greater correspondence in the shape of the input and output signal is required.

References

[1] L. Chua and T. Lin, “Chaos in digital filters”, *IEEE Trans. Circuits and Systems*, Vol 35 pp 648-658, June 1988.

[2] L. Chua and T. Lin, “Fractal pattern in second order nonlinear digital filters”, *Int J. Circuit Theory and Application*, Vol 18, pp 541-550, 1990.

[3] L. Chua and T. Lin, “Chaos and Fractals From Third-Order Digital Filters”, *Int. J. Circuit Theory and Applications*, vol 18, pp 241-255,1990.

[4] C. Mitrovski and Lj. Kocarev, “Nonlinear phenomena in Third order digital filters”, Seville Spain, NDES-1996, pp 405-409, June 1996.

[5] C. Mitrovski, Lj. Kocarev and N. Jonovska, “On a clas of n-th order digital filters operating outside the region of stability”, *Int. Journal of Circuit Theory and Applications*. Vol 26, pp 199-205. February, 1998.