(4,2)-Formal Languages

Violeta Manevska[1], Donco Dimovski[2]

Abstract – **The aim of this paper is to define a (4,2)-semigroup automaton on free (4,2)-semigroup, with special attention on (4,2)-formal languages recognizable by them.**

Keywords – **(4,2)-semigroup, (4,2)-semigroup automaton, (4,2)-language**

I. Introduction

Our goal in writing this talk is to examine a (4,2)-formal language and to proof some properties about them. In that means, we are given an example.

II. (4,2)-Semigroups and (4,2)-Semigroup Automata

Here we recall the necessary definitions and known results. From now on, let B be a nonempty set and let $(B,\cdot)$ be a semigroup, where $\cdot$ is a binary operation.

A semigroup automaton is a triple $(S,(B,\cdot),f)$, where S is a set, $(B,\cdot)$ is a semigroup, and $f:S\times B\to S$ is a map satisfying

$$f(f(s,x),y)=f(s,x,y), \qquad (1)$$

for every $s\in S$, $x,y\in B$.

The set S is called the set of **states** of $(S,(B,\cdot),f)$ and f is called the **transition function** of $(S,(B,\cdot),f)$.

A nonempty set B with the (4,2)-operation $\{\ \}:B^4\to B^2$ is called a **(4,2)-semigroup** iff the following equality

$$\{\{xyzt\}uv\}=\{xy\{ztuv\}\} \qquad (2)$$

is an identity for every $x,y,z,t,u,v\in B$. It is denoted with the pair $(B,\{\})$.

Example 1: Let $B=\{a,b\}$. Then the (4,2)-semigroup $(B,\{\})$ is given by Table 1.

This example of (4,2)-semigroup is generated by an appropriate computer program.

A **(4,2)-semigroup automaton** is a triple $(S,(B,\{\}),f)$ where S is a set, $(B,\{\})$ is a (4,2)-semigroup, and $f:S\times B^2\to S$ is a map satisfying

$$f(f(s,x,y),z,t)=f(s,\{xyzt\}), \qquad (3)$$

for every $s\in S$, $x,y,z,t\in B$.

The set S is called the set of **states** of $(S,(B,\{\}),f)$ and f is called the **transition function** of $(S,(B,\{\}),f)$.

[1]Violeta Manevska, University "St. Clement Ohridski"-Bitola, Faculty of Technical Sciences, Bitola, Ivo Lola Ribar b.b. 7000 Bitola, Macedonia, e-mail: violeta.manevska@uklo.edu.mk

[2]Donco Dimovski, Univesity "Sts. Cyril and Methodus"-Skopje, Faculty of Mathematics and Natural Sciences, Institute of Mathematics, e-mail: donco@iunona.pmf.ukim.edu.mk

Table 1. (4,2)-Semigroup

{ }	
a a a a	(a,a)
a a a b	(a,a)
a a b a	(a,a)
a a b b	(a,a)
a b a a	(a,a)
a b a b	(a,b)
a b b a	(a,a)
a b b b	(a,a)
b a a a	(b,b)
b a a b	(b,b)
b a b a	(b,a)
b a b b	(b,b)
b b a a	(b,b)
b b a b	(b,b)
b b b a	(b,b)
b b b b	(b,b)

2.1^0. Let $(S,(B,\cdot),\varphi)$ be a semigroup automaton. Then $(S,(B,\{\}),f)$ is a (4,2)-semigroup automaton with (4,2)-operation $\{\}:B^4\to B^2$ defined by $\{xyzt\}=(x\cdot y\cdot z,t)$ and the transition function $f:S\times B^2\to S$ defined by

$$f(s,x,y)=f(s,x\cdot y). \qquad ([2])$$

2.2^0. If $S,(B,\{\}),f)$ is a (4,2)-semigroup automaton, then:

i) $(B^2,*)$ is a semigroup, where the operation $*$ is defined by $(x,y)*(u,v)=\{xyuv\}$ for every $(x,y)(u,v)\in B^2$;

(ii) $(S,(B^2,*),\psi)$ is a semigroup automaton, where the transition function $\psi:S\times B^2\to S$ is defined by

$$\psi(s,(x,y))=f(s,x,y). \qquad ([2])$$

Example 2: Let $(B,\{\})$ be a (4,2)-semigroup given by Table 1 from Example 1 and $S=\{s_0,s_1,s_2\}$. A (4,2)-semigroup automaton $(S,(B,\{\}),f)$ is given by Table 2 and the graph in Fig. 1. This example of (4,2)-semigroup automaton is generated by computer.

III. Free (4,2)-Semigroups and (4,2)-Semigroup Automata on Them

Let B be a nonempty set. We define a sequence of sets $B_0,B_1,...,B_p,B_{p+1},...$ by induction as follows:

$B_0=B$. Let B_p be defined, and let A_p be the subset of B_p of all the elements u_1^{2+2s}, $u_\alpha\in B_p$, $s\geq 1$. Define B_{p+1} to be $B_{p+1}=B_p\bigcup A_p\times\{1,2\}$.

Let $\bar{B}=\bigcup\limits_{p\geq 0}B_p$. Then $u\in\bar{B}$ iff $u\in B$ or $u=(u_1^{2+2s},i)$ for some $u_\alpha\in\bar{B}$, $s\geq 1$, $i\in\{1,2\}$.

Table 2. (4,2)-Semigroup

f	(a,a)	(a,b)	(b,a)	(b,b)
s_0	s_1	s_1	s_0	s_1
s_1	s_1	s_1	s_1	s_1
s_2	s_1	s_1	s_2	s_1

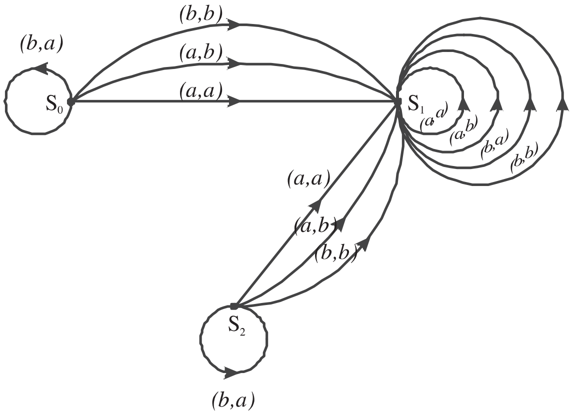

Fig. 1. (4,2)-semigroup automaton

Define a length for elements of $\bar{B}$, i.e. a map $|\;|:\bar{B}\to N$ (N is a set of positive integers) as follows:

1^0 If $u\in B$, then $|u|=1$;

2^0 If $u=(u_1^{2+2s},i)$, then $|u|=|u_1|+|u_2|+\cdots+|u_{2+2s}|$.

By induction on the length we are going to define a map $\varphi:\bar{B}\to\bar{B}$. For $b\in B$, let $\varphi(b)=b$. Let $u\in\bar{B}$and suppose that for each $v\in\bar{B}$ with $|v|<|u|$, $\varphi(v)\in\bar{B}$ and

(1) If $\varphi(v)\neq v$, then $|\varphi(v)|<|v|$;

(2) $\varphi(\varphi(v))=\varphi(v)$.

Let $u=(u_1^{2+2s},i)$. Then, for each α, $\varphi(u_\alpha)=v_\alpha\in\bar{B}$ is defined, $|\varphi(u_\alpha)|\leq|u_\alpha|$ and $\varphi(\varphi(u_\alpha))=\varphi(u_\alpha)$. Let $v=(v_1^{2+2s},i)$.

(i) If for some α, $u_\alpha\neq v_\alpha$, then $|v_\alpha|<|u_\alpha|$, and so, $|v|<|u|$. In this case let $\varphi(u)=\varphi(v)$.

Because $|v|<|u|$, it follows that $\varphi(v)$ is defined, and moreover, (1) and (2) imply that

$$|\varphi(u)|=|\varphi(\nu)|\leq|v|<|u|,\ \varphi(u)\neq u \text{ and}$$
$$\varphi(\varphi(u))=\varphi(\varphi(v)=\varphi(v)=\varphi(u).$$

(ii) Let $u_\alpha=v_\alpha$ for each α. Then $u=v$. Suppose that there is $j\in\{0,1,...,2s\}$ and $r\geq1$, such that $u_{j+v}=(w_1^{2r+2},i)$ for each $v\in\{1,2\}$ and let t be the smallest such j. In this case, let

$$\varphi(u)=\varphi(u_1^tw_1^{2r+2}u_{t+3}^{2s+2},i).$$

Because $|(u_1^tw_1^{2r+2}u_{t+3}^{2s+2},i)|<|u|$ it follows that $\varphi(u)$ is well defined, and moreover, (1) and (2) imply that

$$\varphi(u)\neq u,\ |\varphi(u)|<|u| \quad\text{and}\quad \varphi(\varphi(u))=\varphi(u).$$

(iii) If $\varphi(u)$ cannot be defined by (i) or (ii), let $\varphi(u)=u$. In this case,

$$\varphi(\varphi(u))=\varphi(u)=u \quad\text{and}\quad |\varphi(u)|=|u|.$$

The above discussion and (i), (ii) and (iii) complete the inductive step, and so we have defined a map $\varphi:\bar{B}\to B$. Moreover, we have proved the following:

Lemma:
(a) For $b\in B$, $\varphi(b)=b$;
(b) For each $u\in\bar{B}$, $|\varphi(u)|\leq|u|$;
(c) For $u\in\bar{B}$, if $\varphi(u)\neq u$, then $|\varphi(u)|<|u|$;
(d) For each $u\in\bar{B}$, $\varphi(\varphi(u))=\varphi(u)$. ■

Now, let $Q=\varphi(\bar{B})$. By Lemma (d),

$$Q=\{u|u\in\bar{B},\varphi(u)=u\}.$$

Define a map [ ]: $Q^4\to Q^2$, by $[u_1^4]=(v_1^2)\Leftrightarrow v_i=\varphi(u_1^4,i)$ for each $i\in\{1,2\}$.

Because $u_j\in Q$, it follows that $(u_1^4,i)\in\bar{B}$, and so $\varphi(u_1^4,i)\in Q$ for each $i\in\{1,2\}$. Hence [ ] is well defined.

Theorem: $(Q,[\;])$ is a free (4,2)- semigroup with a basis B. ([1])

Let $S,(B,\{\}),f)$ be a (4,2)-semigroup automaton.

Now, we define a sequence of maps $\psi_0,\psi_1,...,\psi_p$, $\psi_{p+1},...$ for a sequence of sets $B_0,B_1,...,B_p,B_{p+1},...$ by induction as follows:

$\psi_0:B_0\to B_0$ with $\psi_0(b)=b$, for each $b\in B_0$;
$\psi_1:B_1\to B_0$ with $\psi_1(b_1^n,i)=\{b_1^n\}_i$;
$\psi_2:B_2\to B_0$ with $\psi_2(u_1^n,i)=\{\psi_1(u_1)...\psi_1(u_n)\}_i$;
$\vdots$
$\psi_p:B_p\to B_0$ with $\psi_p(u_1^n,i)=\{\psi_{p-1}(u_1)...\psi_{p-1}(u_n)\}_i$;
$\vdots$

Because $\bar{B}=\bigcup\limits_{p\geq0}B_p$, we define a map $\psi:\bar{B}\to B_0$ with $\psi(u)=\psi_p(u)$ for $u\in\bar{B}$ and $|u|\leq p$. Now we will prove that ψ is well defined. If

$$u=(u_1^r(w_1^{2+2s},i_1)(w_1^{2+2s},i_2)u_{r+3}^{2+2t},i),$$
$$v=(u_1^rw_1^{2+2s}u_{r+3}^{2+2t},i)$$

and $\varphi(u)=\varphi(v)$, we have to prove that $\psi(u)=\psi(v)$. We have

$$\begin{aligned}\psi(u)&=\psi_p(u)=\\&=\psi_p(u_1^r(w_1^{2+2s},i_1)(w_1^{2+2s},i_2)u_{r+3}^{2+2t},i)=\\&=\{\psi_{p-1}(u_1)...\psi_{p-1}(u_r)\psi_{p-1}(w_1^{2+2t},i_1)\psi_{p-1}(w_1^{2+2s},i_2)\\&\psi_{p-1}(u_{r+3})...\psi_{p-1}(u_{2+2t})\}_i=\\&=\{\psi_{p-1}(u_1)...\psi_{p-1}(u_r)\{\psi_{p-2}(w_1)...\psi_{p-2}(w_{2+2s})\}_{i_1}...\\&\{\psi_{p-1}(w_1)...\psi_{p-2}(w_{2+2s})\}_{i_2}\psi_{p-1}(u_{r+3})...\psi_{p-1}(u_{2+2t})\}_i=\\&=\{\psi_{p-1}(u_1)...\psi_{p-1}(u_r)\psi_{p-1}(w_1)...\psi_{p-1}(w_{2+2s})\\&\psi_{p-1}(u_{r+3})...\psi_{p-1}(u_{2+2t})\}_i,\end{aligned}$$

Also,

$$\begin{aligned}\psi(v)&=\psi_p(v)=\psi_p(u_1^rw_1^{2+2s}u_{r+3}^{2+2t},i)=\\&=\{\psi_{p-1}(u_1)...\psi_{p-1}(u_r)\psi_{p-1}(w_1)...\psi_{p-1}(w_{2+2s})\\&\psi_{p-1}(u_{r+3})...\psi_{p-1}(u_{2+2t})\}_i.\end{aligned}$$

Hence $\psi(u)=\psi(v)$. On the other hand, $Q=\varphi(\bar{B})$, so it follows that the restriction of ψ on Q is well defined.

Now again, we define a sequence of maps $\tau_0,\tau_1,...,\tau_p$, $\tau_{p+1},...$ for a sequence of sets $B_0,B_1,...,B_p,B_{p+1},...$ by induction as follows:

$\tau_0:S\times B_0^2\to S$ with $\tau_0(s,u,v)=f(s,u,v)$;

$\tau_1:S\times B_1^2\to S$ with

$$\tau_1(s,(u_1^n,i),(v_1^k,j))=f(s,\psi_1(u_1^n,i),\psi_1(v_1^k,j));$$

$\tau_2 : S \times B_2^2 \to S$ with

$$\tau_2(s,(u_1^n,i),(v_1^k,j)) = f(s,\psi_2(u_1^n,i),\psi_2(v_1^k,j));$$

$\vdots$

$\tau_p : S \times B_p^2 \to S$ with

$$\tau_p(s,(u_1^n,i),(v_1^k,j)) = f(s,\psi_p(u_1^n,i),\psi_p(v_1^k,j)).$$

Now we define a map τ for the sequence of maps $\tau_0,\tau_1,...,\tau_p,\tau_{p+1},...$ by $\tau : S \times \bar{B}^2 \to S$, so that $\tau|_{B_p} = \tau_p$ and

$$\tau(s,(u_1^n,i),(v_1^k,j)) = \tau_p(s,(u_1^n,i),(v_1^k,j)) =$$
$$= f(s,\psi_p(u)1^n,i),\psi_p(v_1^k,j)) = f(s,\psi(u_1^n,i),\psi(v_1^k,j)).$$

Because ψ is well defined, it follows that τ is well defined. On the other hand, $Q = \varphi(\bar{B})$ so $\bar{\varphi}$ denotes the map $\bar{\varphi} : S \times Q^2 \to S$ defined by

$$\bar{\varphi}(s,(u_1^n,i),(v_1^k,j)) = \tau(s,(u_1^n,i),(v_1^k,j)) =$$
$$= f(s,\psi(u_1^n,i),\psi(v_1^k,j)).$$

Moreover, $(S,(Q,[\ ]),\bar{\varphi})$ is a (4,2)-semigroup automaton, where $(Q,[\,])$ is a free (4,2)-semigroup with a basis B.

IV. Recognizable (4,2)-Languages

Any subset $L^{(4,2)}$ of the universal language $Q^* = \bigcup_{p\geq 1} Q^p$, where Q is a free (4,2)-semigroup with a basis B, is called a **(4,2)-language (formal (4,2)-language)** on the alphabet B.

A (4,2)-language $L^{(4,2)} \subseteq Q^*$ is called **recognizable** if there exists:

(1) a (4,2)-semigroup automaton $(S,(B,\{\ \}),f)$, where the set S is finite;

(2) an initial state $s_0 \in S$;

(3) a subset $T \subseteq S$ such that

$$L^{(4,2)} = \{w \in Q^* | \bar{\varphi}(s_0,(w,1),(w,2)) \in T\},$$

where $(S,(Q,[\ ]),\bar{\varphi})$ is the (4,2)-semigroup automaton constructed above, for the (4,2)-semigroup automaton $(S,(Q,\{\ \}),f)$.

We also say that the (4,2)-semigroup automaton $(S,(Q,\{\ \}),f)$ **recognizes** $L^{(4,2)}$, or that $L^{(4,2)}$ is **recognized** by $(S,(Q,\{\ \}),f)$.

Example 3: Let $(S,(Q,\{\ \}),f)$ be a (4,2)-semigroup automaton given in Example 2. We construct the (4,2)-semigroup automaton $(S,(Q,[\,]),\bar{\varphi})$ for the (4,2)-semigroup automaton $(S,(Q,\{\ \}),f)$.

A (4,2)-language $L^{(4,2)}$, which is recognized by the (4,2)-semigroup automaton $(S,(Q,[\,]),\bar{\varphi})$, with initial state s_0 and terminal state s_1 is

$$L^{(4,2)} = \{w \in Q^* | w = w_1 w_2 ... w_{2q},$$

where

$$w_l = \begin{cases} (u_1^n,i),\ n \geq 4,\ u_\alpha \in Q \\ (a^* b^*)^* \end{cases} ,l \in \{1,2,...,q\},\ q \geq 3$$

and:

a) If $i = 1$, then:

a1) $(u_1^n,1) = a$, where

$$\psi_{p-1}(u_1)...\psi_{p-1}(u_n) = a(a \cup b)(a^t b^j)^*,$$

a2) $(u_1^n,1) = b$, where

$$\psi_{p-1}(u_1)...\psi_{p-1}(u_n) = b(a \cup b)(a^t b^j)^*;$$

b) If $i = 2$, then

b1) $(u_1^n,2) = a$, where

$$\psi_{p\text{-}1}(u_1)...\psi_{p\text{-}1}(u_n) = (ba)^+ \cup (a(a \cup b)(a^t b^j)^* \backslash (ab)^+),$$

b2) $(u_1^n,2) = b$, where

$$\psi_{p\text{-}1}(u_1)...\psi_{p\text{-}1}(u_n) = (ab)^+ \cup (b(a \cup b)(a^t b^j)^* \backslash (ba)^+),$$

where $t + j = 2k$, $t,j \in \{0,1,2,...\}$, $k \geq 1$, and finally

$$\psi_p(w_1)...\psi_p(w_q) =$$
$$= (ba)^*(bb \cup a(a \cup b))((a \cup b)(a \cup b))^* =$$
$$(ba)^*(aa \cup ab \cup bb)(aa \cup ab \cup ba \cup bb)^*\}.$$

4.1^0. Let $L^{(4,2)}$ be a (4,2)-language on the set B recognized by (4,2)-semigroup automaton $(S,(Q[\ ]),\bar{\varphi})$. Let $(S,(Q,[\ ]),\bar{\varphi})$ be a (4,2)-semigroup automaton eith initial state s_0 and a set of terminal states $T \subseteq S$. Then $\tilde{L}^{(2,1)} \subseteq L^{(4,2)}$ for any language $L^{(4,2)}$, which is recognized by the semigroup automaton $(S,(Q^2,*),\psi)$ with the same initial state and the same set of terminal states, where $\psi : S \times Q^2 \to S$ is a transition function defined by $\psi(s,(u,v)) = \bar{\varphi}(s,u,v)$ and $\tilde{L}^{(2,1)} = \{\tilde{w} | w \in L^{(2,1)}\}$.

Proof: $L^{(4,2)}$ is a recognizable (4,2)-language on the set B by the (4,2)-semigroup automaton $(S,(Q,[\,]),\bar{\varphi})$ with initial state s_0 and a set of terminal states $T \subseteq S$, so

$$L^{(4,2)} = \{w \in Q^* | \bar{\varphi}(s_0,w) \in T\}.$$

By Proposition 2.2^0, $(S,(Q^2,*),\psi)$ is a semigroup automaton. It recognizes a language $L^{(2,1)}$ with a same initial state s_0 and a same terminal states $T \subseteq S$, so it is of the form

$$L^{(2,1)} = \{w \in (Q^2)^* | \psi(s_0,w) \in T\}.$$

Let $w \in L^{(2,1)}$. It follows that $w \in (Q^2)^*$ and $\psi(s_0,w) \in T$. That

$$\bar{\varphi}(s_0,(\tilde{w},2),(\tilde{w},2)) = \bar{\varphi}(s_0,w) = \psi(s_0,w) \in T.$$

Thus $\tilde{w} \in L^{(4,2)}$, i.e. $\tilde{L}^{(2,1)} \subseteq L^{(4,2)}$. ■

4.2^0. Let $L^{(2,1)}$ be a recognizable language on the set B by a semigroup automaton $(S,(B,||\,||),\xi)$ with an initial state s_0 and a set of terminal states $T \subseteq S$, and $(S,(B,\{\ \}),f)$ be a (4,2) semigroup automaton constructed by a semigroup automaton $(S,(B,||\,||),\xi)$. Let $f : S \times B^2 \to S$ be a transition function defined by $f(s,x,y) = \xi(s,x,y)$. Then $L^{(2,1)} \subseteq L^{(4,2)}$, where $L^{(4,2)}$ is a recognizable (4,2) language on the set B by the (4,2)-semigroup automaton $(S,(Q,[\,]),\bar{\varphi})$ with initial state $s_0 \in S$ and a set of terminal states $T \subseteq S$.

Proof: A language $L^{(2,1)}$ is recognizable by a semigroup automaton $(S,(B,||\,||),\xi)$ with an initial state $s_0 \in S$ and a set of terminal states $T \subseteq S$, so

$$L^{(2,1)} = \{w \in B^* | \xi(s_0,w) \in T\}.$$

By Proposition 2.1^0 $(S,(B,\{\ \}),f)$ is a (4,2)-semigroup automaton. We construct a (4,2) semigroup automaton

$(S,(Q,[\;]),\bar{\varphi})$, where $Q = \varphi(\bar{B})$ and $\bar{\varphi} : S \times Q^2 \to S$ is a transition function defined by

$$\bar{\varphi}(s,(y_1^n,i),(v_1^k,j)) = \varphi_p(s,(u_1^n,i),(v_1^k,j)) = \\ = f(s,[\bar{u}_i^n]_i,[\bar{v}_1^k]_i),$$

where

$$\psi_p(u_1^n,i) = [\psi_{p-1}(u_1)...\psi_{p-1}(u_n)]_i = [\bar{u}_1^n]_i,$$
$$\psi_p(v_1^k,j) = [\psi_{p-1}(v_1)...\psi_{p-1}(v_k)]_j = [\bar{v}_1^k]_j$$

It follows that a recognizable (4,2)-language $L^{(4,2)}$ on the set B by (4,2) semigroup automaton $(S,(Q,[\;]),\bar{\varphi})$, with initial state $s_0 \in S$ and a set of terminal states $T \subseteq S$ is of the form

$$L^{(4,2)} = \{w \in Q^* | \bar{\varphi}(s_0,w) \in T\}.$$

Let $w \in L^{(2,1)}$ and $|w| \geq 2$. Then

$$\bar{\varphi}(s_0,(w,1),(w,2)) = \bar{\varphi}(s_0,w) = \xi(s_0,w) \in T.$$

Thus $w \in L^{(4,2)}$, i.e. $L^{(2,1)} \subseteq L^{(4,2)}$. ■

V. Conclusion

The results was given in this paper, are of the scientific interest, because there was defined a (4,2)-languages as a consequence of the generalization of the semigroup automata in case (4,2). Also, here was given the connection between (2,1)-languages and (4,2)-languages..

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