Julia Multitudes and Theirs Computer Design

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Abstract – **In the recent issue are shown the Gaston Julia's multitudes and their computer presentation.**

Keywords – **Gaston Julia, multitude, segment**

I. Introduction

Many physical systems that are deterministic, it means that their future behavior is defined completely of the past condition of the objects, are so sensitive to the beginning conditions that theirs behavior is difficult to predict. The first difficulties in the deterministic approach in defining the condition of more complex system appear when the mathematicians as Cantor, fon Kox, Peano and Julia show geometric curves that are different from the previous. They are characterized as "selfsimilarity" it means that the shape of every little segment from the curve has the shape of the bigger segment. The length could not be easily defined and their size differ from that of the line and is situated between the size of straight line and plane.

The mathematician B. Mandelbrot concludes that many simple mathematical expressions could lead to "Chaotic" nonperiodical functions that although have stable behavior defined from the beginning conditions. Mandelbrot for the first time uses the concept "fractal"-curve, which size by Hausdorf-Bezicovich is higher than the size of the Euclid's space. The term "fractal" comes from the Latin "fractus", that means "irregular or fragmented". Different algorithms are made for generation of graphic computer fractal images. The mathematician M. F. Barnsly by examining the "Julia" multitudes searches different ways for generation of real images. He invented the method of "iteration functional systems" (IFS) that is complex of "iterative affined transformations" that define the connections between the parts of the image.

In the recent issue are shown the Gaston Julia's multitudes (1893-1978) and their computer presentation. [2,3]

II. Mathematical Presentation of the Julia's Multitudes

Let with C mark the Gaus's plane of complex numbers, and with $\bar{C}$ – the Riemann's sphere $C\bigcup\{\infty\}$ [2]. Let R is a rational function:

$$R(x) = P(x)/Q(x), \quad x \in \bar{C}, \tag{1}$$

where P and Q are polynomials that have not common divisors.

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We presume that the function degree of R, $\deg R = \max\{\deg P, \deg Q\}$ is greater than 1. This degree is equal of the number of the proimages of the point x,

$$R^{-1}(x) = \{y \in \bar{C} : R(y) = x\}. \tag{2}$$

The Julia multitude J_R is commonly a multitude of exceptional points of the function $R : R^n(x) = R(...(R(R(x)))...)$, $n = 1, 2, 3, ...$ The addition to the multitude J_R is called Fatu multitude $F_R = \bar{C}\backslash J_R$. The classic definition for the Julia multitude is very comfortable for intuitive learning. That's why we take another definition, more accessible for understanding; it means the *periodic trajectory*. In every $x_0 \in \bar{C}$, the correlation $x_{n+1} = R(x_n)$, $n = 1, 2, ...$, defines certain sequence of points. This sequence is called *positive semitrajectory* of the point x_0 and is marked with $Or^+(x_0)$ [2,3]. When defining the negative semitrajectory could spring up difficulties because of the noncomplex of the reverse image R^{-1}. Although, taking all proimages we put:

$$Or^-(x_0) = \{x \in \bar{C} : R^k(x) = x_0 \text{ for } k = 0, 1, 2, ...\} \tag{3}$$

If $x_n = x_0$ in $Or^-(x_0)$ in known n, could be said that x_0 is periodic point. In this case $Or^-(x_0)$ is called periodic trajectory or circle that is marked with $\gamma = \{x_0, R(x_0), ..., R^{n-1}(x_0)\}$. If n is the smallest natural number having the pointed attribute, than n is called trajectory period.

In the case $n = 1$ there is the equality $R(x_0) = x_0$, it means that x_0 is immovable point of the function R. It is obvious that if x_0 is periodic point of the period n, than x_0 is unmovable point of the function R^n. (There have not to be mixed up the iterations of R and the rank of R, it means $R^n(x) = R \circ R \circ \cdots \circ R(x)$ and $(R(x))^n$.)

For the characterization of the stability of the periodic point x_0 with period n, have to be calculated the derivative. The complex number $\lambda = (R^n)'(x_0)\left(' = \dfrac{d}{dx}\right)$ is called *self meaning* of the point x_0. Using the rule for differencing the complicated function we see that that number is the same for every point from the cycle. The periodic point x_0 is called:

- *supergravitation* $\Leftrightarrow \lambda = 0$,
- *neutral* $\Leftrightarrow |\lambda| = 1$,
- *gravitation* $\Leftrightarrow 0 < |\lambda| < 1$,
- *repulsion* $\Leftrightarrow |\lambda| > 1$.

The Julia multitude J_R could be described with the rational function R. Let P is multitude of all the repulsion periodic points of the function R. If x_0 is arbitrary gravitation unmovable point, than we examine its gravitation zone

$$A(x_0) = \{x \in \bar{C} : R^k(x) \to x_0, \text{ when } k \to \infty\}; \tag{4}$$

$A(x_0)$ consists of these points x, which positive semitrajectories $Or^{+}(x)$ agree in point. This multitude consists of the negative semitrajectory of the point x_0, $Or^{-}(x_0)$. If $\gamma = \{x_0, R(x_0), ..., R^{n-1}(x_0)\}$ is gravitation cycle of the period n, then each of the unmovable points $R^{i}(x_0)$, $i = 0, 1..., n-1$ of the function R^n has its own gravity zone and $A(\gamma)$ is just union of these zones [2].

III. Fundamental Features of the Julia Multitude

1. J_R consists of more than numbered multitude of points.
2. The Julia multitudes of Julia functions R and R^k, $k = 2, 3, ...$ coincide.
3. $R(J_R) = J_R = R^{-1}(J_R)$.
4. For each point $x \in J_R$ its negative semitrajectory $Or^{-}(x_0)$ is continuous in J_R.
5. If γ is gravity cycle of the function R, than $A(\gamma) \subset F_R = \bar{C} \setminus J_R$ and $\partial A(\gamma) = J_R$.

(Here $\partial A(\gamma)$ means the limit of the multitude $A(\gamma)$, it means that $x \in \partial A(\gamma)$, if $x \notin A(\gamma)$ and exists a sequence of points from $A(\gamma)$, agree in point x.)

On fig. 1 and fig. 2 are shown examples for Julia multitudes restricting two, even four different zones of gravity unmovable points.

6. If the Julia multitude has internal points (points $x \in J_R$, such that for known $\varepsilon > 0$, $\{x : |x - \bar{x}| < \varepsilon\} \subset J_R$) then $J_R = \bar{C}$.
7. Such a situation obviously is met rarely and although, one of the examples gives the function $R(x) = ((x - 2)/x)^2$
8. If $\bar{x} \in J_R$ and $\varepsilon < 0$, then exists whole n, in which $R^n(J^n) = J_n$.

From the features results of that each rational image has big reserve of repulsion points. That's why the Julia multitude is not changing when the image R is acting, but the dynamic of J_R is chaotic. The fifth feature shows the calculation algorithm for receiving images of the multitude J_R. Unfortunately the negative semitrajectory of the point $\bar{x} \in J_R$ usually is not distributed equally in the Julia multitude. That's why we need more complex algorithms for solving each time which of the branches of the tee structure $Or^{-}(\bar{x})$ have to be taken for most effective building of the image. Such algorithms are made and used for creation of our images. The sixth feature shows that in many of the cases the multitude J_R has to be fractal. For example, if R has more than 2 gravity unmovable points $a, b, c, ...$, then

$$\partial A(a) = J)R = \partial A(b) = J_R = \partial A(c) = \cdots , \quad (5)$$

it means that the limits of all gravity zones coincide. So, if R has 3 or 4 gravity unmovable points, then J_R consists of three sided or four sided points according to the gravity zones [2,3].

IV. Methods for Receiving of Computer Images of the Julia Multitudes

There are two different methods for receiving computer images of the Julia multitudes. One of them is based on the fifth feature and the other on the sixth feature. None of the methods has special advantages. In some cases the first method works better, in others the second. There are many cases when the two methods work perfect. But there are whole class of Julia multitudes for which is very difficult to be received satisfying images (if it is possible to receive any images). This class consists of Julia multitudes that limit parabolic regions, it means that correspond to images with parabolic periodic point.

A. *Method of reverse iteration*

If a rational image R is given and is known one periodic repulsion point $\bar{x} \in J_R$, then the feature (5) permits to be calculated

$$J_R^n = \{x \in \mathbf{C} : R^k(x) = \bar{x} \text{ for given } k \leq n\}. \quad (6)$$

B. *Modified method for reverse iteration*

The strategy is the following: over J_R is put rectangular grid with small size β. After this, for each cell B from the grid, have to be stopped the use of points from it for reverse iteration if certain number of $N_{\max}$ points in B are already have been used. It is usually that the optimal choice of β and $N_{\max}$ depends of μ_R and of the parameters of the computer image as the resolution of the used system. Therefore an iteractive and adaptive algorithms are necessary [2].

On fig. 1 and fig. 2 are shown two figures of images received from the Julia multitudes by the modified method for reverse iteration.

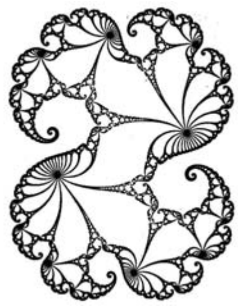

Fig. 1.

On fig. 1 is shown:

$$x \to (1+\varepsilon)\lambda x + x^2 \quad (7)$$
$$\lambda = e^{\frac{2\pi i}{20}}, \quad x_0 = 0, \quad \varepsilon = 0.001$$

On fig. 2 is shown image of the Julia multitude with parabolic unmovable point for: [1,2,4]

$$x \to \lambda x + x^2 \quad (8)$$
$$\lambda = e^{\frac{2\pi i}{20}}, \qquad x_0 = 0.$$

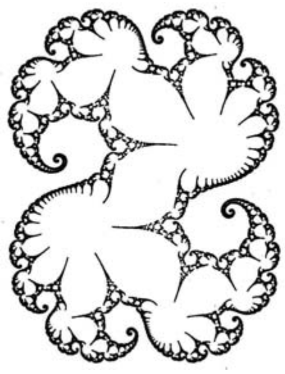

Fig. 2.

V. Additional Remarks

1. It is possible to be received interest pictures and with the help of two different colors – black and white. It is possible the use of color till $K = 200$;
2. The picture that is received is symmetric according the beginning of the coordinate system;
3. The time for calculation could be reduced two times because of the symmetry of the calculation process;
4. All the points that not incline to infinity after K steps, will be colorized in black, including the points that lay in regions of repulsion of other attractors, if exist such.

VI. Conclusion

The fractal image is infinite series of iteration and in the reverse iteration, the resolution of the display is not important, as independent from the level of zooming of the image, the level of the details is not changing.

A conclusion could be made that the method for signal and image compression through their fractal presentation is one of the most perspective and interesting in this field. It may be attention for additional scientist researches with purpose of creating new real program instrumentation for mass programs in effective and cheap computer multimedia systems. For example for the purposes of the television with high density – HDTV, such results will be necessary because of its future global use.

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