Hausdorff LC Filters

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Abstract - **This paper presents a method for synthesis of two kinds of inverse filters with Hausdorff-type transfer characteristic. The filter frequency characteristics are determined and a comparison with Chebyshev inverse filters is given.**
Keywords - **approximation, polynomial, synthesis, inverse LC filter, Hausdorff, Chebyshev, frequency characteristic**

I. Introduction

In modern filter theory the synthesis is performed by appropriate characteristic function approximation [2], [3]. Hausdorff filters are implemented by "shifted" Delta-function approximation [1] with the same form line as of an ideal characteristic function with Hausdorff polynomial, given on Fig. 1 and Equation (1):

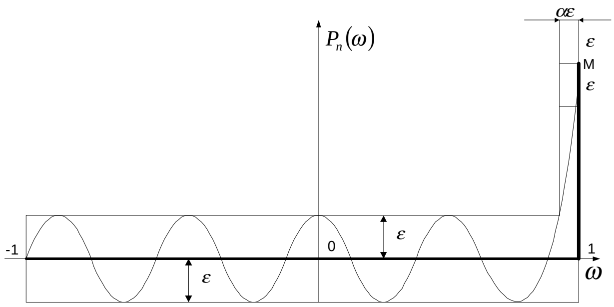

Fig. 1

$$P_n(\omega) = \varepsilon T_n\left(\frac{2\omega}{2-\alpha\varepsilon}\right) = \varepsilon T_n\left(\frac{\omega}{1-\alpha\varepsilon/2}\right). \qquad (1)$$

In this equation ε (Hausdorff dimension) is the best approximation with algebraic polynomial of "shifted" Delta-function in Hausdorff metrics [1], T_n is Chebyshev polynomial of first order and n degree; α is parameter, $\omega=2\pi f$ is frequency and the product $\alpha\varepsilon$ determines function steepness in an interval approximating the transition between filter pass-band (PB) and filter stop-band (SB). Given the filter order n and pass-band ripple DA [dB], the Hausdorff dimension ε and the product $\alpha\varepsilon$ could be found from the following equations (2) and (3), [4]:

$$\varepsilon = \sqrt{10^{0.1DA}-1}\,; \qquad (2)$$

$$\alpha\varepsilon = 2\frac{\mathrm{ch}\left[\frac{1}{n}\mathrm{Ach}\left(\frac{1}{\varepsilon}\right)\right]-1}{\mathrm{ch}\left[\frac{1}{n}\mathrm{Ach}\left(\frac{1}{\varepsilon}\right)\right]+1}. \qquad (3)$$

Hausdorff filters transfer function takes the form:

$$|A(\omega)| = \frac{1}{\sqrt{1+\varepsilon^2 T_n^2\left(\frac{\omega}{1-\alpha\varepsilon/2}\right)}}. \qquad (4)$$

As $\alpha\varepsilon$ values are in the interval (0,1) from the equation (4) can be seen that Chebyshev polynomial argument is divided by a positive number smaller than 1. That's why the Hausdorff filter transmission function appears to be "scale-shrunk" compared to Chebyshev ones with a coefficient of $(1-\alpha\varepsilon/2)$. Hausdorff and Chebyshev filters transmission functions are shown on Fig. 2. Both functions contain identical variation in pass-band and identical steepness out of it when the difference for the normalized frequency (cut-off frequency in this case) is equal to $\alpha\varepsilon/2$ [5].

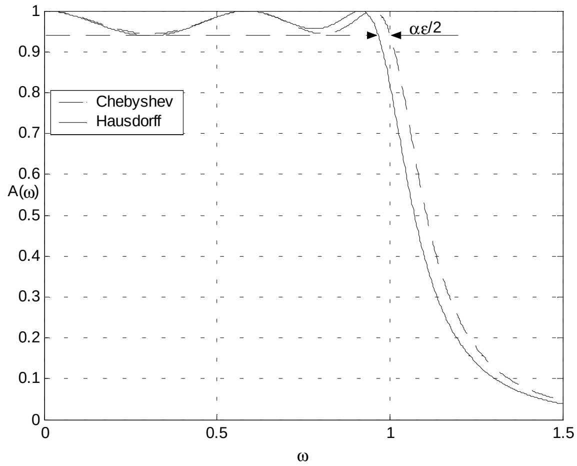

Fig. 2

II. Inverse Hausdorff Filters (IHF)

A. Inverse Hausdorff Filters from A-type

An inverse filter transmission function comes out through low-frequency filter-prototype transmission function transfer into high-frequency transmission function and the reciprocal value of argument is taken $\omega = 1/\omega$:

$$|A_{nA}(\omega)| = \sqrt{\frac{\varepsilon^2 T_n^2\left[\frac{1}{\omega(1-\alpha\varepsilon/2)}\right]}{1+\varepsilon^2 T_n^2\left[\frac{1}{\omega(1-\alpha\varepsilon/2)}\right]}}. \qquad (5)$$

As can be seen from the last equation Chebyshev polynomial argument is multiplied by the expression $(1-\alpha\varepsilon/2)$. This comes to transmission function "scale-stretching" along the ω axis with a coefficient $(1-\alpha\varepsilon/2)$. Inverse Hausdorff and Chebyshev filters will have identical steepness and attenuation in stop-band, as ordinary (non-inverse) Hausdorff filters when filter order n and characteristics value k for limit frequency ω_c, are equal (Fig. 3).

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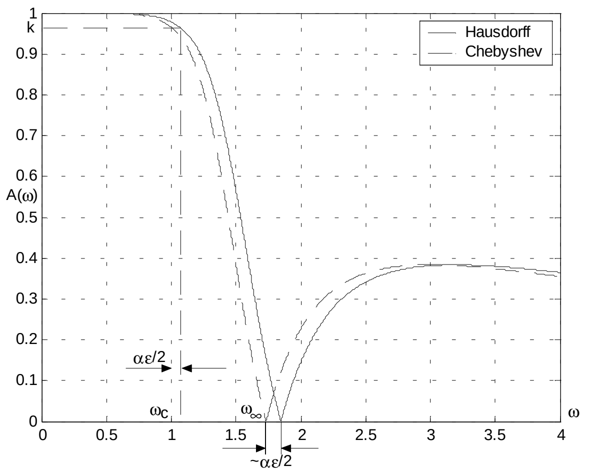

Fig. 3

The cut-off frequency shift ω_c is not acceptable in terms of filter design. As for inverse Hausdorff filters this can be overcome with the help of modern filter synthesis theory [2], [3]. An IHF is to be designed with limit frequency equal to a preliminary specified value of transfer function *k*, the pass band frequencies proportional to the product $\alpha\varepsilon$, as shown on Fig.4.

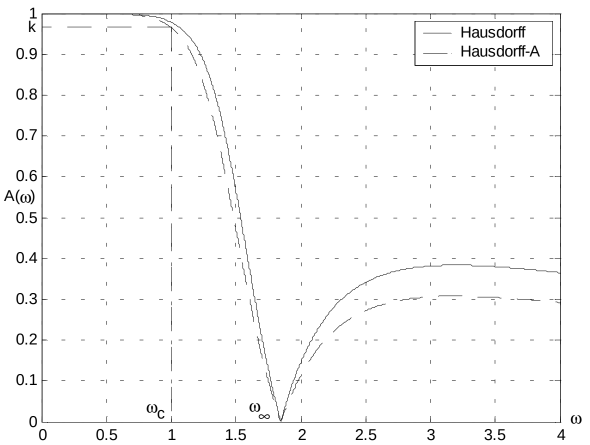

Fig. 4

Filter order *n*, cut-off frequency ω_c, transfer function attenuation for it *DA* in [dB], stop frequency ω_s and the product $\alpha\varepsilon$ are to be known for transfer function parameters specification. Transfer function limit frequency value is specified like in (2):

$$k = \sqrt{10^{0.1DA} - 1}\,. \tag{6}$$

The product $\alpha\varepsilon$ can be specified by calculating transfer function value of inverse Chebyshev filter from the same order in stop frequency:

$$DS_{ch} = 10\log\left[k^2\cosh^2\left(n\arccos h\frac{\omega_s}{\omega_c}\right) + 1\right]. \tag{7}$$

Then ε of the low-pass filter-prototype can be calculated [6]:

$$\varepsilon = \frac{1}{\sqrt{10^{0.1DS} - 1}}\,. \tag{8}$$

As it was mentioned in the beginning, the plain Hausdorff and Chebyshev filters when pass-band ripple is equal, the characteristics steepness is also equal, i.e. when the relation ω_s/ω_c is equal they will have equal attenuation. Therefore, the established value for will be equal to the value of Hausdorff dimension of the low-pass Hausdorff filter-prototype. When it is known, $\alpha\varepsilon$ can be determined from the equation (3).

The requirement that the cut-off frequency must remain the same leads to a change in stop-band maximums, which can be seen on Fig 4. The attenuation that IHF will have with $\omega_s(1-\alpha\varepsilon/2)/\omega_c$ as value of argument is calculated:

$$DS_{1A} = 10\log\left\{k^2\mathrm{ch}^2\left[n\mathrm{Ach}\frac{\omega_s(1-\alpha\varepsilon/2)}{\omega_c}\right] + 1\right\}. \tag{9}$$

From the established value IHF equivalent ripple ε_1 can be determined:

$$\varepsilon_{1A} = \frac{1}{\sqrt{10^{0.1DS_{1A}} - 1}}\,. \tag{10}$$

Then the attenuation that IHF will have for stop-band ω_s will be:

$$DSH_A = 10\log\frac{\varepsilon_{1A}^2\mathrm{ch}^2\left(n\mathrm{Ach}\dfrac{1}{1\text{-}\alpha\varepsilon/2}\right)}{1+\varepsilon_{1A}^2\mathrm{ch}^2\left(n\mathrm{Ach}\dfrac{1}{1\text{-}\alpha\varepsilon/2}\right)}\,. \tag{11}$$

B. Inverse Hausdorff filters from B-type

From the point of view of the synthesis it would be interesting to form an inverse filter transfer function taking in equation (1) reciprocal value of the whole expression containing the Chebyshev polynomial argument:

$$|A_{nB}(\omega)| = \sqrt{\frac{\varepsilon^2T_n^2\left(\dfrac{1-\alpha\varepsilon/2}{\omega}\right)}{1+\varepsilon^2T_n^2\left(\dfrac{1-\alpha\varepsilon/2}{\omega}\right)}}\,. \tag{12}$$

In this case, in contrast to (5), the argument ω is divided by $(1-\alpha\varepsilon/2)$. This leads to "scale-compressing" of transfer function (Fig. 5) along the axis ω a coefficient of $(1-\alpha\varepsilon/2)$.

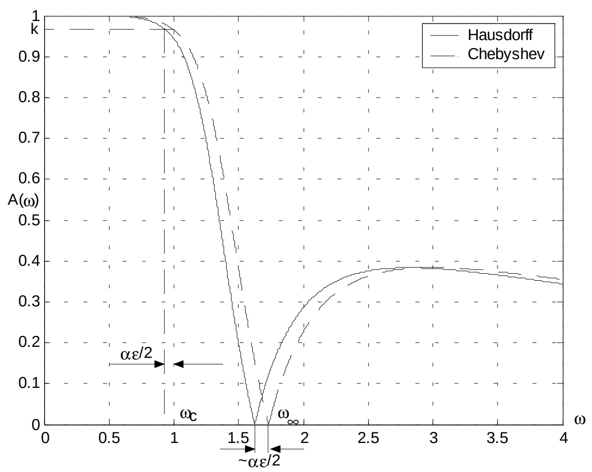

Fig. 5

The transfer function we want to realize is shown on Fig. 6. This type filter parameters can be defined as those for A-type. The product $\alpha\varepsilon$ is established from the equations (6), (7), (8) and (3) and the expressions (9), (10) and (11) take the form:

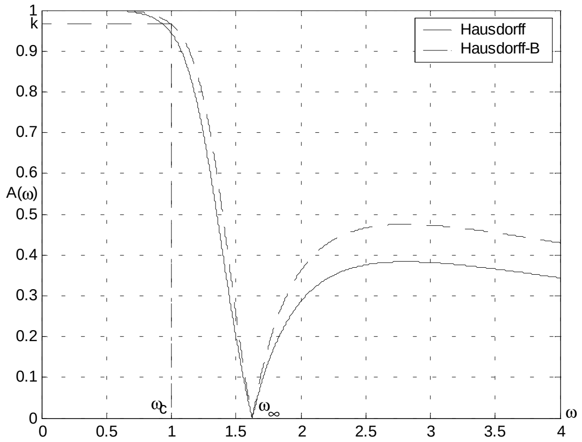

Fig. 6

$$DS_{1B} = 10\log\left\{k^2 \mathrm{ch}^2\left[n\mathrm{Ach}\frac{\omega_s}{\omega_c(1-\alpha\varepsilon/2)}\right]+1\right\}; \quad (13)$$

$$\varepsilon_{1B} = \frac{1}{\sqrt{10^{0.1DS_{1B}}-1}}; \quad (14)$$

$$DSH_B = 10\log\frac{\varepsilon_{1B}^2 \mathrm{ch}^2\left[n\mathrm{Ach}(1-\alpha\varepsilon/2)\right]}{1+\varepsilon_{1B}^2 \mathrm{ch}^2\left[n\mathrm{Ach}(1-\alpha\varepsilon/2)\right]}. \quad (15)$$

III. Synthesis basics

The characteristics and transfer function are represented as relation of three polynomials $e(s)$, $p(s)$ и $q(s)$ of complex frequency $s = j\omega$. The polynomial $e(s)$ is Hurwitz strict polynomial and its zeros ω_i represent the filter own frequencies and these of $p(s)$ - the extreme frequencies $\omega_{\infty i}$, for which the transfer function has infinite attenuation. Calculating two of polynomials usually solves the synthesis task and the third is defined by Feldkeller equation:

$$e(s)e(-s) = p(s)p(-s) + q(s)q(-s). \quad (16)$$

The zeros of *e(s)* and *p(s)* can be found as follows:

For A-type:

$$\omega_i = \frac{1}{\left(1-\frac{\alpha\varepsilon}{2}\right)(\sigma_i + j\Omega_i)}; \quad (17)$$

$$\omega_{\infty i} = \frac{j}{\left(1-\frac{\alpha\varepsilon}{2}\right)\cos\left(\frac{2i-1}{n}\frac{\pi}{2}\right)}. \quad (18)$$

For B-type:

$$\omega_i = \frac{1-\frac{\alpha\varepsilon}{2}}{\sigma_i + j\Omega_i}; \quad (19)$$

$$\omega_{\infty i} = \frac{j\left(1-\frac{\alpha\varepsilon}{2}\right)}{\cos\left(\frac{2i-1}{n}\frac{\pi}{2}\right)}, \quad (20)$$

where:

$$\sigma_i = -\sin\left(\frac{2i-1}{n}\frac{\pi}{2}\right)\mathrm{sh}\left[\frac{1}{n}\mathrm{Ash}\left(\frac{1}{\varepsilon_{1A,B}}\right)\right]; \quad (21)$$

$$\Omega_i = \cos\left(\frac{2i-1}{n}\frac{\pi}{2}\right)\mathrm{ch}\left[\frac{1}{n}\mathrm{Ash}\left(\frac{1}{\varepsilon_{1A,B}}\right)\right], (i = 1 \div n). \quad (22)$$

Elements value calculation is carried by a method described in [17] via transformation of the variable *s* into a new variable *z*:

$$z = 1+\frac{\omega_c^2}{s^2}. \quad (23)$$

Basing on this method two computer programs APPROX and LC are given in [3] where the filter is calculated. On input data load frequencies defined from the equations (18) and (20) must be introduced.

Through the described synthesis method two low-pass Hausdorff filters from third order (*n*=3) and cut-off frequency $f_c = 10\mathrm{kHz}$, stop frequency $f_s = 15\mathrm{kHz}$, $\alpha\varepsilon = 0.1239$, attenuation 0.3dB for cut-off frequency, normalized input and output resistance of 1Ω were calculated. Schematic diagrams 7 and 8 of the filters are shown below:

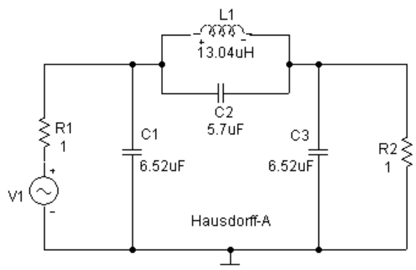

Fig. 7

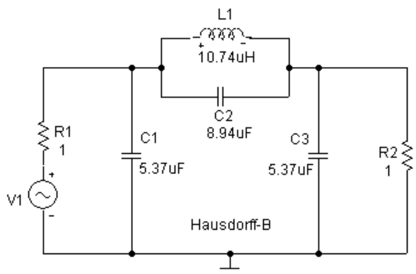

Fig. 8

IV. Inverse Hausdorff filters frequency characteristics

A. Magnitude (amplitude)-frequency characteristic

On Fig 9 are shown magnitudes responses of the two filters compared to that of inverse Chebyshev filter with the same input data.

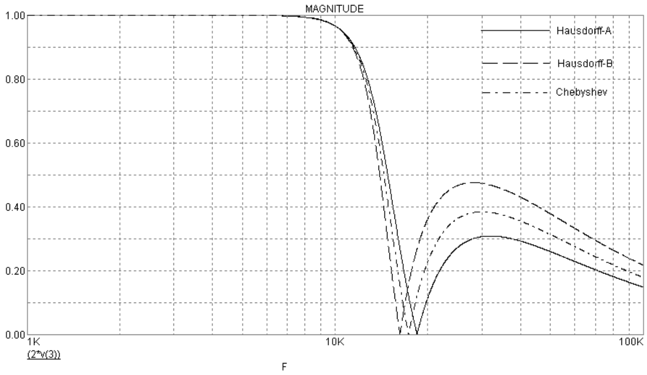

Fig. 9

The IHF from A-type have less steepness in interval (ω_c, ω_s) compared to the inverse Chebyshev. It is because their extreme frequencies are $(1-\alpha\varepsilon)^{-1}$ times higher than those of Chebyshev's (18). They have greater attenuation in stop-band. The extreme frequencies to IHF B-type are $(1-\alpha\varepsilon)$ times less than Chebyshev's, which defines the higher characteristic steepness in the interval (ω_c, ω_s) and less attenuation in stop-band.

B. Phase-Frequency Characteristic

Equations (17) and (19) solution defines the polynomials $e(s)$. Represented as rational function after a substitution $s = j\omega$, they expand into real e_R and imaginary e_I polynomials. The same operation is applied to the polynomials $p(s)$, defined from (18) and (20). Phase-frequency characteristic is:

$$\varphi(\omega) = \arctan\frac{p_I}{p_R} - \arctan\frac{e_I}{e_R}. \qquad (24)$$

On Fig 10 phase-frequency characteristics of the two types inverse Hausdorff filters are shown compared with that of Chebyshev's with the same input data.

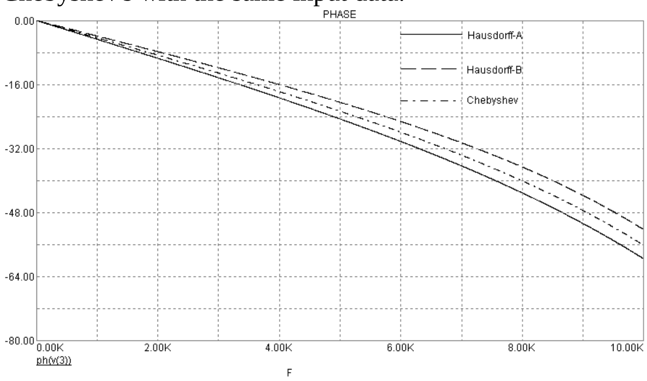

Fig. 10

The A-type inverse Hausdorff filter has more linear characteristics in pass-band compared to that of Chebyshev's and the B type's characteristic is more non-linear.

C. Group time-delay (GTD)

It can be defined from the equation:

$$t_{gr} = \mathrm{Re}\left[\frac{1}{e(s)}\frac{de(s)}{ds} - \frac{1}{p(s)}\frac{dp(s)}{ds}\right]. \qquad (25)$$

On Fig 11 group time delay of the two types inverse Hausdorff filters are shown compared to that of inverse Chebyshev filter with the same input data

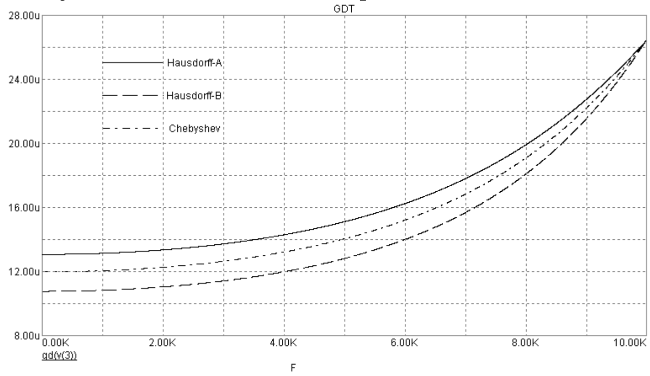

Fig. 11

The A-type inverse Hausdorff filter has more linear characteristic in pass-band GTD compared to that of Chebyshev's and the B-type is more non-linear.

IV Conclusion

We may say as a conclusion that the two types of inverse filters are complimentary of one another. Compared to Chebyshev's, the first type shortcomings are second type advantages. This leads to greater opportunities in filter design. The frequency characteristics specificity is defined from Hausdorff dimension. This means that the two types Hausdorff filters are unique and without analogous using other type of approximation.

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