New First-Order Very Low Sensitivity Narrow-Band Hypercomplex Digital Filter Sections

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Abstract - **In this paper a new strategy to obtain very low sensitivity narrow-band orthogonal hypercomplex coefficients first-order IIR digital filter sections with canonic number of elements is developed. It is shown that the new filters behave much better in a limited word-length environment in comparison with other known structures of this type.**

Keywords – **complex filters, orthogonal hypercomplex digital filter sections, sensitivity**

I. INTRODUCTION

As a rule hypercomplex numbers are defined as an expansion of complex numbers [1]. Hamilton's quaternion belongs to the group of hypercomplex numbers and has its main application in the field of image processing and computer graphics as a coordinate transform of 3-D images. They have been also employed by the aerospace engineering, where their convenient representation of rotation has proved useful in such matters as stabilization and altitude control. Shutte and Wenzel proved that Hamilton's quaternions and biquaternions are not well suited for the purpose of digital signal processing (DSP) and in Ref. [2] the modified version as "reduced biquaternions (RB)" is offered. Some authors call RB "bicomplex numbers". Several research studies concerning applications of bicomplex numbers to DSP have been reported. Computational efficiency and stability criterion of digital filters with bicomplex coefficients are investigated in Ref. [9], [8]. Colour image filters based on hypercomplex convolution are defined and utilized for autocorrelation and cross-correlation of color image processing in Ref. [10]. Hypercomplex filters also make use of image recognition, smoothing the colour image components, design of two-dimensional transfer functions with applications in image processing of both grey scale and colour images.

The filters with RB coefficients can reduce the order of the filters to 1/4 of the one with real coefficients, and to 1/2 of one with complex coefficients, which is one of the most significant advantages of using hypercomplex numbers. Reduction of the number of multipliers and additions by efficient algorithms is another advantage.

In many applications high accuracy filter realizations with heavily quantized coefficients are required and they are achieved by structures with lower sensitivity. In this paper some new first-order narrow-band hypercomplex digital filter sections with very low coefficients sensitivity are developed and investigated.

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II. PROPERTIES OF HYPERCOMPLEX NUMBERS

A quaternion may be represented in hypercomplex form as follows:

$$q = a + bi + cj + dk \,, \tag{1}$$

where a, b, c and d are real numbers, while i, j, and k are orthogonal complex operators which obey the following rules:

$$\begin{gathered} i^2 = j^2 = k^2 = ijk = -1 \\ ij = k \quad jk = i \quad ki = j \quad ji = -k \quad kj = -i \quad ik = -j \,. \end{gathered} \tag{2}$$

Regarding the arithmetic lows of a quaternion, addition of two quaternions is commutative and associative, and multiplication is associative but not commutative, which make them not applicable to DSP systems. To avoid this problem reduced biquaternion (RB) has been proposed, which is derived as follows: a and b in Eq. (1) are expanded to complex numbers and c and d are set to zero. This modification is equivalent to the expansion of each element of a complex number to a complex number, i.e. a quaternion is a complex number with complex real and imaginary parts. On account of that RB is also called bicomplex number:

$$\begin{aligned} A = A_S + iA_V &= (A_1 + jA_2) + i(A_3 + jA_4) = \\ &= A_1 + jA_2 + iA_3 + kA_4 \end{aligned} \tag{3}$$

where A_1, A_2, A_3 and A_4 are real numbers, j is the imaginary unit with $j^2 = -1$ and i is the vector unit with $i^2 = -1$. The first two terms A_S are called "scalar part" and the last two A_V - "vector part". The properties of the imaginary units for RB are as follows:

$$\begin{gathered} i^2 = j^2 = -1 \quad k^2 = 1 \\ ij = ji = k \quad jk = kj = -i \quad ki = ki = -j \,. \end{gathered} \tag{4}$$

Two types conjugate of A can be defined – the vector conjugate:

$$A^+ = A_S - iA_V \tag{5}$$

and the complex conjugate:

$$A^* = A_s^* + iA_v^* = (A_1 - jA_2) + i(A_3 - jA_4) \,. \tag{6}$$

Applying Euler's formula for the complex exponential generalized to hypercomplex form any quaternion q may be represented in polar form as:

$$q = |q| e^{\mu\Phi} \,. \tag{7}$$

μ and Φ are referred as the eigenaxis and eigenangle of *q*, respectively. Φ is analogous to the argument of a complex number, but is unique only in the range $[0, \pi]$, because the value greater than π can be reduced to this range by reversing of eigenaxis. μ is a unit (pure) quaternion and identifies the direction in three-space of the hypercomplex number's vector part. The requirement for μ is $|\mu| = 1$.

Hereafter in this publication each coefficient and internal signal of hypercomplex digital filters will be encoded as a bicomplex number.

III. Hypercomplex sections derivation procedure outline

If the variable *z* in a *N*-order real coefficients digital transfer function $H_R(z)$ is substituted by

$$z = ze^{-j\theta} = z(\cos\theta - j\sin\theta), \quad (8)$$

the complex coefficients transfer function $H_C(z) = H(e^{-j\theta})$ will be obtained. $H_C(z)$ may be easily presented by two 2*N*-order real coefficients transfer functions:

$$H_C(\hat{z}) = H_{R1}(z) + jH_{R2}(z), \quad (9)$$

where "R" denotes real, "C" – complex and "$\hat{z}$" – complex variable.

If $H_R(z)$ is a low-pass (LP) type, $H_{R1}(z)$ and $H_{R2}(z)$ will be of band-pass (BP) type only, while high-pass (HP) $H_R(z)$ will produce band-stop (BS) type transfer functions as well. $\theta = \pi/2$ substituted in Eq. (8) brings to

$$z = -jz \quad (10)$$

and respectively $H_C(z) = H(-jz)$ will be complex transfer function called "orthogonal".

In this work the idea proposed by Okabayashi and Takahashi [7] is used in order to derive a first-order hypercomplex (bicomplex) orthogonal transfer function $H_{HC}(z)$ presented by fourth-order real coefficients transfer functions $H_1(z)$, $H_2(z)$, $H_3(z)$ and $H_4(z)$:

$$\begin{aligned} H_{HC}(z) &= H_1(z) + jH_2(z) + iH_3(z) + kH_4(z) = \\ &= H_{C1}(\hat{z}) + iH_{C2}(\hat{z}). \end{aligned} \quad (11)$$

Scalar part $H_{C1}(\hat{z})$ and vector part $H_{C2}(\hat{z})$ have complex coefficients and "HC" marks hypercomplex. The coefficients of $H_{HC}(z)$ are bicomplex numbers represented by Eq. (3).

If "*j*" is replaced by "*i*" in Eq. (9), bicomplex coefficients transfer function will be retrieved:

$$H_{HC}(z) = H_{R1}(z) + iH_{R2}(z), \quad (12)$$

but the coefficients of $H_{R1}(z)$ and $H_{R2}(z)$ will be still real because $H_{HC}(z)$ has coefficients whose scalar and vector components are with zero imaginary parts:

$$A = (A_1 + j0) + i(A_3 + j0) = A_1 + iA_3 \quad A_1, A_3 \in R. \quad (13)$$

This corresponds to the orthogonal transformation $z = -iz$ applied on real transfer function $H_R(z)$ and leads to the orthogonal bicomplex function Eq. (12).

Then, the complex frequency transformation according to *j* imaginary unit

$$z = \hat{z}e^{-j\theta} \quad (14)$$

is applied on $H_{R1}(z)$ and $H_{R2}(z)$ in order to get:

$$H_{HC}(\hat{z}) = H_{C1}(\hat{z}) + iH_{C2}(\hat{z}). \quad (15)$$

$H_{C1}(\hat{z})$ and $H_{C2}(\hat{z})$ have complex coefficients because now the imaginary parts of scalar and vector components are not zero:

$$A = (0 + jA_2) + i(0 + jA_4) = j(A_2 + iA_4) \quad A_2, A_4 \in R. \quad (16)$$

Orthogonal form of the frequency transformation Eq. (14)

$$z = \hat{z}e^{-j\frac{\pi}{2}}, \quad (17)$$

applied on $H_{R1}(z)$ and $H_{R2}(z)$, will make possible the presentation of $H_{C1}(\hat{z})$ and $H_{C2}(\hat{z})$ in the terms of real transfer functions:

$$\begin{aligned} H_{C1}(\hat{z}) &= H_1(z) + jH_2(z), \\ H_{C2}(\hat{z}) &= H_3(z) + jH_4(z). \end{aligned} \quad (18)$$

Thus the representation of a bicomplex orthogonal transfer function by four real transfer functions Eq. (11) is achieved.

IV. First-order orthogonal hypercomplex filter sections derivation

After intensive search among the most often used first-order LP filter sections it was found that MHNS-section (Fig.1a) is rather appropriate prototype having canonic number of elements. Its transfer function is:

$$H_R^{MHNS}(z) = \frac{1-\alpha}{2}\frac{1+z^{-1}}{1-\alpha z^{-1}}. \quad (19)$$

In [3] a method of circuit transformation is proposed permitting also to obtain orthogonal complex filters with canonic number of elements. Applying this method on the section in Fig.1a, the complex orthogonal structure from Fig.1b is obtained [5]. Having two inputs and two outputs as well, this section is able to realize four real coefficients transfer functions at its different outputs which are equal in couples as follows:

$$H_{R1}^{MHNS}(z)=H_{RR}(z)=H_{II}(z)=\frac{(1-\alpha)}{2}\cdot\frac{1-\alpha z^{-2}}{1+\alpha^2 z^{-2}}, \qquad (20)$$

$$H_{R2}^{MHNS}(z)=H_{RI}(z)=-H_{IR}(z)=\frac{(1-\alpha^2)}{2}\cdot\frac{z^{-1}}{1+\alpha^2 z^{-2}}. \qquad (21)$$

They are obtained after orthogonal transformation Eq. (10) is applied on Eq. (19) and all four are of BP type. They can be considered as real and imaginary parts respectively of the complex transfer function:

$$H_{C}^{MHNS}(z)=H_{R1}^{MHNS}(z)+jH_{R2}^{MHNS}(z). \qquad (22)$$

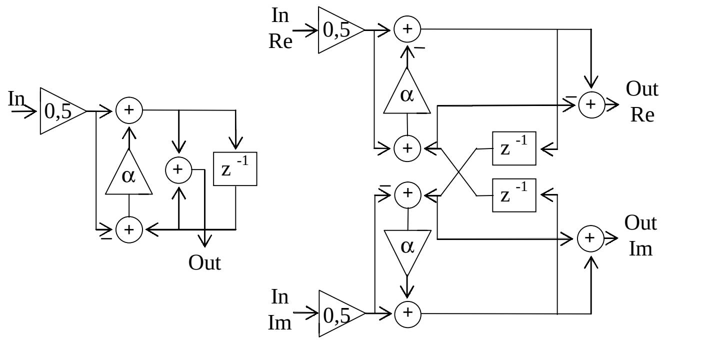


Fig.1. (a) First-order LP MHNS-prototype real sections; (b) MHNS-based orthogonal BP complex section

Similar transformation, producing orthogonal bicomplex filter structures with canonic number of elements, could easily be developed. Realization by real elements (multipliers, delays and additions) is expected to be rather complicated circuit having four inputs and the same number of outputs. Therefore sixteen fourth-order transfer functions will be carried out four by four equal with $\pm$ sign.

According to the above presented method we apply orthogonal frequency transformation Eq. (17) on $H_{R1}^{MHNS}(z)$ and $H_{R2}^{MHNS}(z)$. As a result the following orthogonal fourth-order real coefficients transfer functions are obtained:

$$H_{1}^{MHNS}(z)=\frac{1-\alpha}{2}\cdot\frac{1+\alpha^3 z^{-4}}{1-\alpha^4 z^{-4}}, \qquad (23)$$

$$H_{2}^{MHNS}(z)=\frac{\alpha(1-\alpha^2)}{2}\cdot\frac{z^{-2}}{1-\alpha^4 z^{-4}}, \qquad (24)$$

$$H_{3}^{MHNS}(z)=\frac{\alpha^2(\alpha^2-1)}{2}\cdot\frac{z^{-3}}{1-\alpha^4 z^{-4}}, \qquad (25)$$

$$H_{4}^{MHNS}(z)=\frac{\alpha^2-1}{2}\cdot\frac{z^{-1}}{1-\alpha^4 z^{-4}}. \qquad (26)$$

They form the bicomplex transfer function $H_{HC}^{MHNS}(z)$.

It was shown in [5] that in the case of narrow-band LP filter (pole near z=1) the MHNS-structure (Fig. 1a) has very high coefficients sensitivity. Using Nishihara's method [4] of sensitivity minimization through coefficient conversion, the section shown in Fig.2a and named LS11 is derived. Its transfer function is:

$$H_{R}^{LS11}(z)=\frac{\beta}{2}\frac{1+z^{-1}}{1-(1-\beta)z^{-1}}. \qquad (27)$$

Following the procedure applied on the MHNS-section, we derive the complex structure in Fig.2b with real coefficients transfer functions:

$$H_{R1}^{LS11}(z)=H_{RR}(z)=H_{II}(z)=\frac{\beta}{2}\cdot\frac{1-(1-\beta)z^{-2}}{1+(1-\beta)^2 z^{-2}}, \qquad (28)$$

$$H_{R2}^{LS11}(z)=H_{RI}(z)=-H_{IR}(z)=\frac{\beta}{2}\cdot\frac{(2-\beta)z^{-1}}{1+(1-\beta)^2 z^{-2}}, \qquad (29)$$

being parts of the orthogonal complex transfer function.

The orthogonal bicomplex transfer function's real parts are

$$H_{1}^{LS11}(z)=\frac{\beta}{2}\cdot\frac{1+(1-\beta)^3 z^{-4}}{1-(1-\beta)^4 z^{-4}}, \qquad (30)$$

$$H_{2}^{LS11}(z)=\frac{(1-\beta)\left[1-(1-\beta)^2\right]}{2}\cdot\frac{z^{-2}}{1-(1-\beta)^4 z^{-4}}, \qquad (31)$$

$$H_{3}^{LS11}(z)=\frac{(1-\beta)^2\left[(1-\beta)^2-1\right]}{2}\cdot\frac{z^{-3}}{1-(1-\beta)^4 z^{-4}}, \qquad (32)$$

$$H_{4}^{LS11}(z)=\frac{(1-\beta)^2-1}{2}\cdot\frac{z^{-1}}{1-(1-\beta)^4 z^{-4}}. \qquad (33)$$

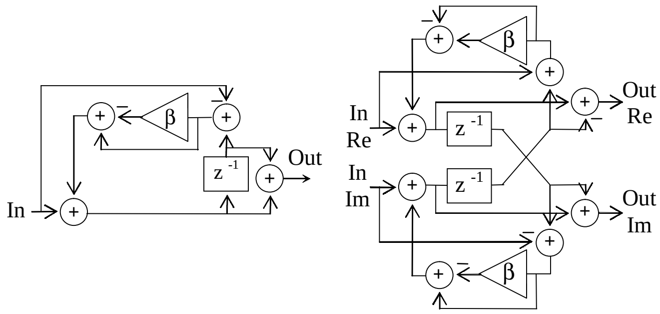


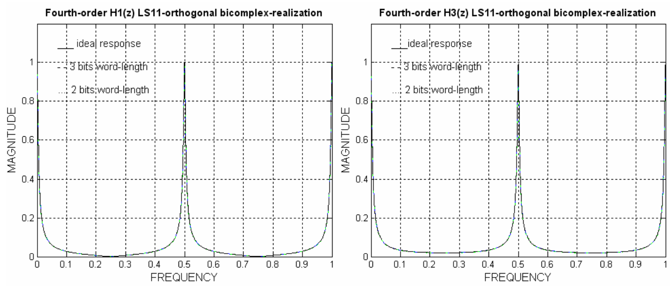
Fig.2. (a) First-order LP LS11-prototype real sections; (b) LS11-based orthogonal BP complex section

V. SENSITIVITY INVESTIGATIONS

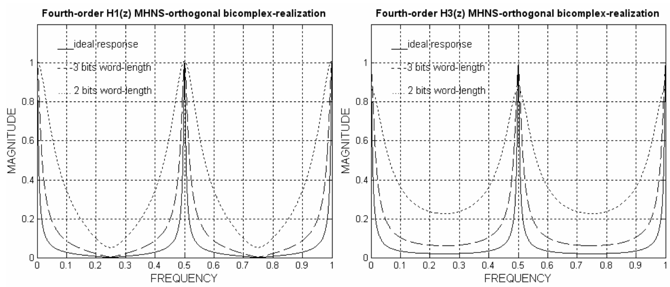
In this section the coefficients sensitivity of all three type (real, complex and bicomplex) filter sections will be investigated. The case of narrow-band orthogonal filter which is the most likely in practice is achieved for α=0.99 (β=0.01). Canonic sign-digit (SD) code representation and fixed point arithmetic were used and the word-length was changed from infinite (ideal case) to 2 bits only.

Under this conditions the real prototype-sections (Fig.1a and 2a) have been investigated. We have shown [5], [6] that coefficients sensitivity of LS11 LP first-order filter section is about hundred times lower that this of MHNS-section. It was shown also that the low-sensitivity properties of the LP-prototype circuits are directly inherited in the complex coefficients orthogonal structures.

In order to verify the reported results a number of computer simulation were conducted with respect to the bicomplex filter sections, derived in this publication. All eight transfer functions (23)-(26) for MHNS-based and (30)-(33) for LS11-based section were simulated for the same pole disposition near the unit circle. Poles $p_{1,2} = \pm 0{,}99$ and $p_{3,4} = \pm j0{,}99$ are achieved for $\alpha = 0{,}99$ (MHNS-based) and $\beta = 0{,}01$ (LS11-based) structures. In Fig.3a and 3b some of the results concerning $|H_1|$ and $|H_3|$ of the MHNS- and the LS11-based structures are shown for different coefficient word-lengths. It is seen that the LS11-structure has a magnitude response almost coinciding with the ideal one even when the word-length is reduced to 2 bits only. The MHNS-based structure response (Fig. 3b) is considerably changed when the word-length is only limited to 3 bits. The similar results were obtained also for $|H_2|$ and $|H_4|$.



(a)



(b)

Fig. 3. Magnitude responses of the first-order bicomplex orthogonal sections for different word-lengths: (a) LS11-based for β=0.01; (b) MHNS-based for α=0.99

VI. CONCLUSIONS

A method for design of hypercomplex (bicomplex) coefficients first-order orthogonal filter sections was proposed in this publication. Second-order BP real coefficients transfer functions, composing a first-order complex function, were transformed into fourth-order real coefficients transfer functions after orthogonal frequency transformation. The first-order bicomplex transfer function can be presented by them which reduce four times the order in comparison with the real one.

As narrow-band orthogonal filters are most often used, the developed transfer functions were investigated on these terms for different word-length of the coefficients.

It was experimentally demonstrated that the magnitude response of the LS11 low sensitivity hypercomplex section does not change considerably even when two bits of SD-code is used, while the magnitude response of MHNS-structure is completely destroyed when the word-length is reduced to three bits. These properties, including sensitivity of the LP-prototype structures will be inherited by the new bicomplex structures. Many new sections can be derived and investigated following the proposed approach.

The presented method for design of hypercomplex sections is also appropriate for second-order filter section's derivation which will permit cascade filter realizations.

The new hypercomplex structures are suitable for realization of high quality narrow-band orthogonal filters. Additionally, the possibility for simplification of the circuits and further parallelism will make them very attractive for telecommunications and other DSP applications and will ensure a considerable reduction of the complexity and the cost of the equipment.

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