Scattering Parameters of Exponential Transmission Line

Zlata Cvetkovic[1]

Abstract– **Exponential lossless transmission line as two port network is observed in the paper. Its *ABCD* parameters are determined using Picard-Carson's method. Conversion of *ABCD* parameters to scattering parameters is also done. Scattering parameters are plotted using program package Mathematica 3.0.**

Keywords–**Scattering parameters, *ABCD* parameters, Exponential transmission line, Picard –Carson method.**

I. Introduction

Scattering parameters are important in microwave design because they are easier to be measured and to work with at high frequencies than other kinds of two port network parameters [1].

In the paper [2] formulas for conversions between various networks matrices are presented. While four of these matrices, *Z*, *Y*, *h* and *ABCD*, relate voltage and current at the ports, the other two, *S* and *T*, relate wave quantities. The equations developed in [2] are valid for complex and unique impedances on input and output ports.

Relationships presented in paper [3] between different two port network matrices are valid for homogeneous transmission line.

Author of the paper [4] comments relationships between various two port network matrices presented in paper [3] and in other presented papers and points out that results depend on the chosen reference impedance.

The scattering parameters of a lossless, exponential transmission line are studied in detail both in frequency and time domains [5]. Time domain scattering parameters are obtained by taking the inverse Laplace's transform of their corresponding functions in frequency domain. By taking the causality condition into consideration, the time domain scattering parameters in a rapid convergence power series are cast. Each term of the power series represents a signal component generated by the exponential line when the signal travels around trip.

A new approach for the time domain simulation of transients on a dispersive and lossy transmission line terminated with active devices is presented in [6]. The method combines the scattering matrix of an arbitrary line and the nonlinear causal impedance functions at the loads to derive expressions for the signals at the near and far ends. A time domain formulation is proposed using the scattering matrix representation. The algorithm assumes that dispersion and loss models for the transmission lines are available and that the frequency dependence is known.

Conversion from *ABCD* parameters to scattering parameters is performed in this paper. The scattering parameters for lossless exponential transmission line are determined using given expressions. *ABCD* parameters of transmission line are calculated using Picard-Carson's method.

II. Scattering Parameter Formulation

Any linear two port network can be described as a set of scattering parameters which relate two reflected and two incident waves. These waves are variables, which depend on the total currents and voltages at the two ports.

Scattering parameters are defined as,

$$b_1 = S_{11}\ a_1 + S_{12}\ a_2\,, \tag{1}$$

$$b_2 = S_{21}\ a_1 + S_{22}\ a_2\,, \tag{2}$$

where a_1, a_2, b_1 and b_2 are defined as the incident and reflected waves from Port 1 and Port 2, respectively. S_{ij}, for $i, j = 1,\ 2$, are the scattering parameters.

In generally, they tell us how much power "comes back" or "comes out" when we "throw power at" a network. They also contain phase shift information.

S_{11} and S_{22} are simply the forward and reverse reflection coefficients with the opposite port terminated with reference impedances, which are equal to the characteristic impedances of the exponential line at left and right sides, respectively. S_{12} and S_{21} are the forward and reverse gains assuming reference impedances equal to characteristic impedance of the exponential line at left and right sides, respectively.

The signal flow graph in Fig. 1 gives the situation for the scattering parameter interpretation in voltages.

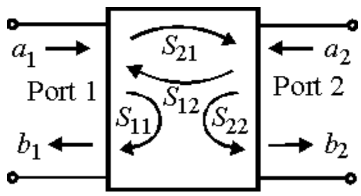

Fig. 1. Two port network represented with scattering parameters.

ABCD parameters for a two port network shown in Fig. 2 are written as

$$U_1 = A\,U_2 - B\,I_2\,, \tag{3}$$

$$I_1 = C\,U_2 - D\,I_2\,. \tag{4}$$

A relation between the *ABCD* parameters of a lossless exponential transmission line and the associated scattering parameters can be derived.

[1] Zlata Z. Cvetkovic is with the Faculty of Electronic Engineering, University of Nis, Beogradska 14, 18 000 Nis, Serbia, Yugoslavia, E-mail: zlata@elfak.ni.ac.yu

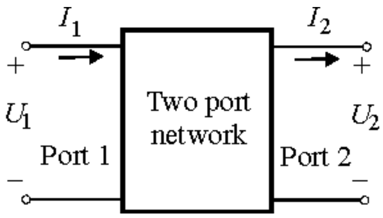

Fig. 2. A general two port network.

If we assume that reference impedances at Port 1 and Port 2 are pure real and equal to characteristic impedance of the exponential line at left and right sides, R_{01} and R_{02}, respectively, we can start from relations

$$a_j = \sqrt{R_{0j}}\, I_{ji} \tag{5}$$

and

$$b_j = \sqrt{R_{0j}}\, I_{jr}\,, \tag{6}$$

where R_{0j} is the reference impedance for the j-th port [2], [3..

I_{ji} and I_{jr} are the incident and reflected currents for the j-th port, so

$$I_j = I_{ji} - I_{jr} \tag{7}$$

and

$$U_j = U_{ji} + U_{jr} \tag{8}$$

where U_{ji} and U_{jr} are the incident and reflected voltage at the j-th port.

After substitution of relations (5), (6), $U_{ji} = I_{ji} \cdot R_{0j}$ and $U_{jr} = I_{jr} \cdot R_{0j}$ into (7) and (8) we obtain

$$I_j = \frac{1}{\sqrt{R_{0j}}}\left(a_j - b_j\right) \tag{9}$$

$$U_j = \sqrt{R_{0j}}\left(a_j + b_j\right). \tag{10}$$

These equations are used as the starting equations for conversion from *ABCD* parameters to scattering parameters.

Substituting (9) and (10) into (3) and (4), we get the following expressions for the scattering parameters:

$$S_{11} = \frac{A R_{02} + B - C R_{01} R_{02} - D R_{01}}{A R_{02} + B + C R_{01} R_{02} + D R_{01}}\,, \tag{11}$$

$$S_{12} = \frac{2\left(AD - BC\right)\sqrt{R_{01} R_{02}}}{A R_{02} + B + C R_{01} R_{02} + D R_{01}}\,, \tag{12}$$

$$S_{21} = \frac{2\sqrt{R_{01} R_{02}}}{A R_{02} + B + C R_{01} R_{02} + D R_{01}} \tag{13}$$

and

$$S_{22} = \frac{- A R_{02} + B - C R_{01} R_{02} + D R_{01}}{A R_{02} + B + C R_{01} R_{02} + D R_{01}}\,. \tag{14}$$

Unlikely *ABCD* parameters, scattering parameters are dependent upon the source and load impedances.

III. $ABCD$ PARAMETERS OF DISTRIBUTED NETWORKS

To describe the voltage and current behaviour on the nonuniform transmission line, we have Telegraph's equations:

$$\frac{\mathrm{d}U(s,\ x)}{\mathrm{d}x} = -Z(x,s)I(s,\ x) \tag{15}$$

and

$$\frac{\mathrm{d}I(s,\ x)}{\mathrm{d}x} = -Y(x,s)U(s,\ x)\,, \tag{16}$$

where $Z(x,\ s) = r(x) + l(x)s$ is serial impedance and $Y(x,\ s) = g(x) + c(x)s$ is the shunt admittance per unit length of the distributed network. $r(x)$, $l(x)$, $g(x)$ and $c(x)$ are unit length resistance, inductance, admittance, and capacitance, which are functions of position x, for a nonuniform line.

In this paper a successive approximation method, Picard-Carson's method is used to solve transmission line equations (15) and (16). This method is a powerful method in getting a power series solution for the distributed network [7]. Picard–Carson's method solves ordinary differential equations by an iterative sequence

$$U_n = U_0 - \int_0^x Z(x,s) I_{n-1}(x,s)\,\mathrm{d}x \quad \text{and} \tag{17}$$

$$I_n = I_0 - \int_0^x Y(x,s) U_{n-1}(x,s)\,\mathrm{d}x\,, \tag{18}$$

for n=1,2,3,…, where U_0 and I_0 are the voltage and current at the input port, $x = 0$, Fig. 3.

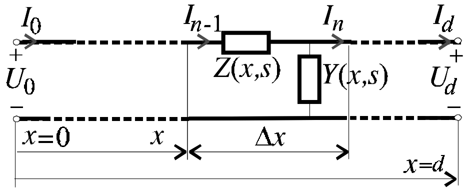

Fig. 3. Transmission line and model of elementary line length.

Since the terms inside the integrals are continuous and bounded, the sequences will converge to the true solutions

$$U(s,x) = \lim_{\Delta x \to 0} U_n(s,x) \tag{19}$$

and

$$I(s,x) = \lim_{\Delta x \to 0} I_n(s,x). \tag{20}$$

Equations (17) and (18) may be written in the form of

$$U(s,x) = U_0 - \int_0^x Z(x,s) I(s,x)\,\mathrm{d}x\,, \tag{21}$$

$$I(s,x) = I_0 - \int_0^x Y(x,s) U(s,x)\,\mathrm{d}x\,. \tag{22}$$

Using Picard–Carson's method we can show these solution in the form of two port parameters [7],

$$U(s,x) = U_0 \sum_{n=0}^{\infty} \zeta_{2n} - I_0 \sum_{n=0}^{\infty} \zeta_{2n+1} \tag{23}$$

$$I(s,x) = U_0 \sum_{n=0}^{\infty} \psi_{2n+1} - I_0 \sum_{n=0}^{\infty} \psi_{2n} \,. \tag{24}$$

From equation (23) and (24), the *ABCD* parameters are:

$$A(x) = \sum_{n=0}^{\infty} \psi_{2n}, \quad B(x) = \sum_{n=0}^{\infty} \zeta_{2n+1}\,, \tag{25}$$

$$C(x) = \sum_{n=0}^{\infty} \psi_{2n+1} \text{ and } D(x) = \sum_{n=0}^{\infty} \zeta_{2n}\,. \tag{26}$$

To start iteration, we can choose

$$\zeta_0 = 1 \quad \text{and} \quad \psi_0 = 1\,, \tag{27}$$

as initial values. The other terms in the summations are found by evaluating the following integrals iteratively:

$$\zeta_n = \int_0^x Z\, \psi_{n-1}\, \mathrm{d}\,x \tag{28}$$

and

$$\psi_n = \int_0^x Y\, \zeta_{n-1}\, \mathrm{d}\,x\,. \tag{29}$$

IV. *ABCD* PARAMETERS OF LOSSLESS TRANSMISSION LINE

For example of Picard–Carson's method we consider a lossless exponential transmission line of known characteristic impedance,

$$Z_C(x,s) = Z_{C0}\, \mathrm{e}^{k\,x}\,, \tag{30}$$

with $k = \dfrac{1}{d} \ln \dfrac{Z_{Cd}}{Z_{C0}}$ and $Z_{C0} = \sqrt{\dfrac{l_0}{c_0}}$,

where Z_{C0} and Z_{Cd} are the characteristic impedances of the exponential line at the left (source) and right (load) sides, respectively. l_0 and c_0 are the inductance and capacitance per unit length. $Z(x,s) = l_0 s\, \mathrm{e}^{k\,x}$ and $Y(x,s) = c_0 s\, \mathrm{e}^{-k\,x}$ are serial impedance and the shunt admittance per unit length of the line.

Starting from (27) and using equations (28) and (29), a few first terms of series are

$$\zeta_1(x) = \frac{l_0 s}{k}\left(\mathrm{e}^{k\,x} - 1\right),\ \psi_1(x) = \frac{c_0 s}{k}\left(\mathrm{e}^{-k\,x} - 1\right),$$

$$\zeta_2(x) = \frac{l_0 c_0 s^2}{k^2}\left(1 - \left(\mathrm{e}^{k\,x} - k\,x\right)\right),$$

$$\psi_2(x) = -\frac{l_0 c_0 s^2}{k^2}\left(1 - \left(\mathrm{e}^{-k\,x} + k\,x\right)\right),$$

$$\zeta_3(x) = \frac{l_0^{\,2} c_0 s^3}{k^3}\left(2 - 2\,\mathrm{e}^{k\,x} + \left(1 + \mathrm{e}^{k\,x}\right)k\,x\right),$$

$$\psi_3(x) = -\frac{l_0 c_0^{\,2} s^3}{k^3}\left(2 - \mathrm{e}^{-k\,x}\left(2 + \left(1 + \mathrm{e}^{k\,x}\right)k\,x\right)\right),$$

$$\zeta_4 = \frac{l_0^{\,2} c_0^{\,2} s^4}{k^4}\left(3 + \frac{1}{2}\left(-6\,\mathrm{e}^{k\,x} + 2\left(2 + \mathrm{e}^{k\,x}\right)k\,x\right) + k^2 x^2\right)$$

$$\psi_4 = \frac{l_0^{\,2} c_0^{\,2} s^4}{k^4}\left(3 + \frac{1}{2}\left(-6\,\mathrm{e}^{-k\,x} - 2\left(2 + \mathrm{e}^{-k\,x}\right)k\,x\right) + \mathrm{e}^{k\,x} k^2 x^2\right)$$

$$\zeta_5 = \frac{l_0^{\,3} c_0^{\,2} s^5}{k^5}\left(-6 + \frac{1}{2}\left(12\,\mathrm{e}^{k\,x} - 6\left(1 + \mathrm{e}^{k\,x}\right)k\,x\right) + \left(\mathrm{e}^{k\,x} - 1\right)k^2 x^2\right)$$

$$\psi_5 = \frac{l_0^{\,2} c_0^{\,3} s^5}{k^5}\left(6 + \frac{1}{2}\left(-12\,\mathrm{e}^{-k\,x} - 6\left(1 + \mathrm{e}^{-k\,x}\right)k\,x\right) + \left(\mathrm{e}^{k\,x} - 1\right)k^2 x^2\right)$$

$\vdots$

After substitution these terms into (18) and (19), we obtain the *ABCD* parameters.

V. RESULTS FOR SCATTERING PARAMETERS

In this paper, we consider an exponential transmission line, which is used as impedance transformer from $Z_{C0} = 100\,\Omega$ to $Z_{Cd} = 400\,\Omega$ [5]. If the length of transmission line is $d = 0.5\,\mathrm{m}$ then k will be $k = 0.5753641$.

For chosen transmission line *ABCD* parameters are

$$A(d) = \sum_{n=0}^{\infty} \psi_{2n}(d) = 1 + \psi_2 + \psi_4 + \psi_6 + \cdots \approx$$
$$\approx 1 + 0.11(s\tau)^2 + 0.0023(s\tau)^4 + 0.000019(s\tau)^6\,,$$

$$B(d) = \sum_{n=0}^{\infty} \xi_{2n+1}(d) = \zeta_1 + \zeta_3 + \zeta_5 + \cdots \approx$$
$$\approx Z_{C0}\left[0.579 s\tau + 0.024(s\tau)^3 + 0.0003(s\tau)^5 + 1.79\times10^{-6}(s\tau)^7\right]$$

$$C(d) = \sum_{n=0}^{\infty} \psi_{2n+1}(d) = \psi_1 + \psi_3 + \psi_5 + \psi_7 + \cdots \approx$$
$$\approx \frac{1}{Z_{C0}}\left[0.43 s\tau + 0.018(s\tau)^3 + 0.00023(s\tau)^5 + 1.34\times10^{-6}(s\tau)^7\right]$$

and

$$D(d) = \sum_{n=0}^{\infty} \xi_{2n}(d) = 1 + \zeta_2 + \zeta_4 + \zeta_6 + \cdots \approx$$
$$\approx 1 + 0.14(s\tau)^2 + 0.0029(s\tau)^4 + 0.000025(s\tau)^6\,.$$

where τ is time rise,

$$\tau = \sqrt{l_0 c_0}\,. \tag{31}$$

Comparing to the first term in the given series, the high order terms can be neglected. In order to simplify the expression, we focus on the first three terms of A and D parameters and the first two terms of B and C parameters. After substitution of given values of *ABCD* parameters into (11)–(14), we obtain approximate values for the scattering parameters in Laplace's domain. The time domain scattering parameters $S_{i\,j}(t)$ are the inverse Laplace's transforms of the frequency domain scattering parameters. This gives

$$S_{i\,j}(t) = L^{-1}\left[S_{i\,j}(s)\right], \qquad (i, j = 1,\ 2\,). \tag{32}$$

where L^{-1} represents the inverse Laplace's transform.

The inverse Laplace's transform of scattering parameters is done in program package Mathematica 3.0 in this paper. Scattering parameters are plotted using the same program. All results are calculated for impedance ratio $Z_{Cd} / Z_{C0} = 4$.

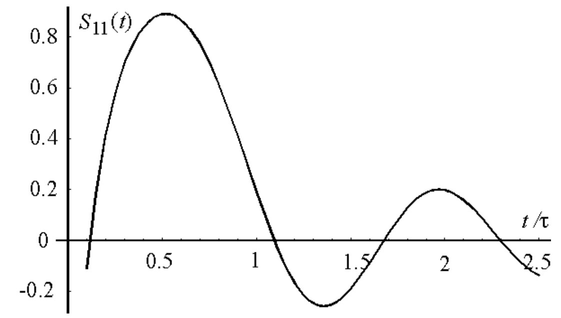

Fig. 4. Time domain scattering reflection coefficient $s_{11}(t)$.

Figs. 4 and 5 show the time domain input and output reflection coefficients $s_{11}(t)$ and $s_{22}(t)$, respectively. These magnitudes are always less than 1. The steady state values of $s_{11}(t)$ and $s_{22}(t)$ are zero.

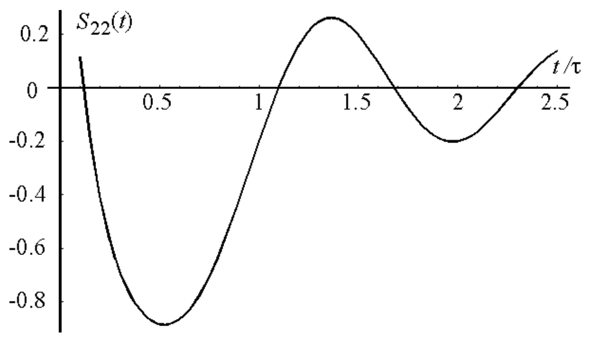

Fig. 5. Time domain scattering reflection coefficient $s_{22}(t)$.

In the Fig. 6 the reverse transmission coefficient $s_{12}(t)$ is plotted and forward transmission coefficient $s_{21}(t)$ in the Fig.7.

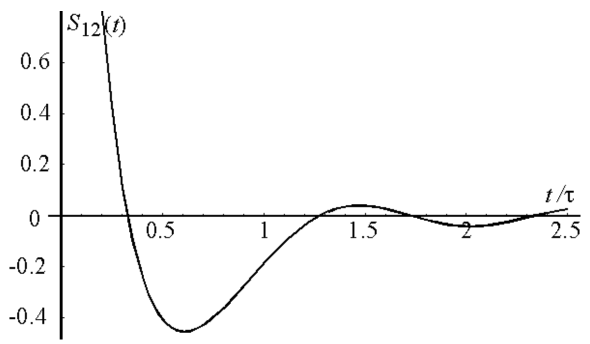

Fig. 6. Time domain scattering coefficient $s_{12}(t)$.

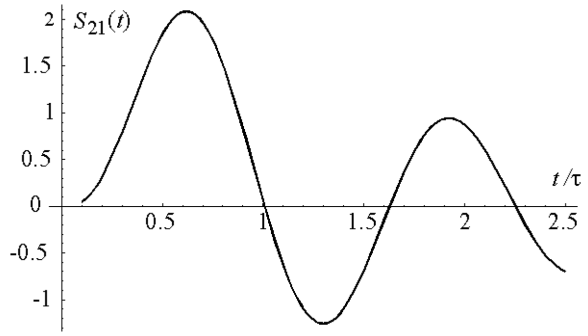

Fig. 7. Time domain scattering coefficient $s_{21}(t)$.

We can conclude that according to (25) and (26), if we carry out the iteration for infinite times, scattering parameters will thus contain infinite number of terms a power of s. These parameters are directly related to the moments of the function. In this paper we only make use of first four poles. So these results depend on number of chosen terms and on used program package.

VI. Conclusion

The Picard–Carson's method of determining the parameters of the general distributed two port network is used as a practical method for obtaining *ABCD* parameters of the two port network of nonuniform transmission line in this paper. The scattering parameters are obtained using formulas, which are obtained by conversions from *ABCD* parameters. *ABCD* parameters are approximated by first a few terms (of first four poles) of a series. Obtained scattering parameters are plotted using program package Mathematica 3.0.

References

[1]http://eesof.tm.agilent.com/docs/iccap2002/MDLGBOOK/1MEASUREMENTS/3VNA/3SPAR/1SparBasics_1.pdf

[2] Markovic, V., *Modelling of Noise of Microwave Transistors*, Memoir, Nis, 2002.

[3] Frickey, D. A., “Conversions Between S, Z, Y, h, ABCD, and T Parameters which are Valid for Complex Source and Load Impedance”, *IEEE Trans. Microwave Theory Tech.*, Vol. 42, No. 2, pp. 205–211, 1994.

[4] Marks, R. B., and Williams, D. J., “Comments on ‘Conversions Between S, Z, Y, h, ABCD, and T Parameters which are Valid for Complex Source and Load Impedance’ ”, *IEEE Trans. Microwave Theory Tech.*, Vol. 43, No. 4, pp. 914–915, 1995.

[5] Hsue, C. W., “Time-Domain Scattering Parameters of an Exponential Transmission line”, *IEEE Trans. Microwave Theory Tech.*, Vol. 39, No. 11, pp. 1891–1895, 1991.

[6] Schutt-Aine, J. E., and Mittra, R., “Scattering Parameter Transient Analysis of Transmission lines Loaded with Nonlinear Terminations”, *IEEE Trans. Microwave Theory Tech.*, Vol. 38, No 8, pp. 1023-1030,

[7] Ghausi, S. M., and Kelly, J. J., *Introduction to Distributed-Parameter Networks*, Holt, Rinehart and Winston, Inc., New York, 1968.