On Relation Between Minimum Variance and Sliding Mode Equivalent Control Concepts

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Abstract - **The paper investigates the connection existing between the minimum variance control and the discrete-time sliding mode equivalent control. It is shown that the minimum variance control represents the counterpart of the discrete-time equivalent control for the input-output based sliding mode control design approaches.**

Keywords - **Variable structure control systems, Discrete-time sliding mode control**

I. Introduction

Variable structure control (VSC) systems [1] are well-known and well-studied class of nonlinear systems, wherein sliding mode is of particular interest. It happens when a control forces a system state to move along a predefined sliding surface in spite of the actions of external disturbances and parameter perturbations. This is accomplished by a high frequency control signal whose switching is guided by a function also known as the switching function. The switching function, equalized with zero, determines the equation of the above mentioned sliding surface. When the system is in sliding mode, the control can be replaced by the so-called equivalent control [2], a very powerful mean for the analysis of system dynamics. If a system state is on a sliding surface, the equivalent control will ensure a system motions towards the steady-state.

The discretization process and the control algorithm implementation by using a digital signal processor produce a quasi-sliding mode [3] and a discrete-time equivalent control [4]. Regardless of the use of continuous- or discrete-time sliding mode techniques, the equivalent control is based on the system modeling in the state space. For the input-output based control approaches, the combinations of minimum variance and sliding mode controls seem to be more appropriate [5-7].

The aim of this paper is to show that the minimum variance control corresponds to the discrete-time equivalent control for the case when only plant input and output signals are measured to form the control law. As we can see later, this link is settled by the implementation of the techniques for equivalent control derivation on some modified system state-space model.

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The paper is organized as follows. In Section II and Section III, the brief descriptions of the discrete-time equivalent control and the minimum variance control are given, respectively. Section IV presents the procedure which proves that the minimum variance control is really discrete-time equivalent control analog for input-output based VSC systems. This approach is referred to as the input-output based equivalent control concept. In Section V, the illustrative example of the second order system demonstrates the established connections between the minimum variance control and the equivalent control, and explains why the equivalent control algorithm, described in Section IV, can not be used for treating the VSC problems based on measuring of input-output signals.

II. Discrete-time Equivalent Control

Let us consider a discrete-time state-space model of a single-input-single-output plant of the *n*-th order:

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{\Phi}\mathbf{x}(k) + \boldsymbol{\gamma} u(k), \\ y(k) &= \mathbf{d}\mathbf{x}(k) = x_1(k), \end{aligned} \tag{1}$$

where: $\mathbf{x}(k) = [x_1(k) \quad x_2(k) \quad ... \quad x_n(k)]^{\mathrm{T}}$ is a vector of state coordinates, $u(k)$ is a control input, $y(k)$ is a plant output, $\mathbf{\Phi} = \lfloor \phi_{ij} \rfloor_{n\mathrm{x}n}$ and $\boldsymbol{\gamma} = \lfloor \gamma_i \rfloor_{n\mathrm{x}1}$. We suppose that a system is autonomous, i.e. a reference signal is equal to zero. Notice that $\bullet(k) = \bullet(kT)$ with T as a sampling period. Let the switching function be:

$$s(k) = \boldsymbol{\sigma}\mathbf{x}(k), \quad \boldsymbol{\sigma} = [\sigma_0 \quad \sigma_1 \quad ... \quad \sigma_{n-1}], \tag{2}$$

with the coefficients of $\boldsymbol{\sigma}$ forming the Jury's polynomial.

By substituting Eq. (1) in Eq. (2), someone gets:

$$s(k+1) = \boldsymbol{\sigma}\mathbf{\Phi}\mathbf{x}(k) + \boldsymbol{\sigma}\boldsymbol{\gamma} u(k) \ . \tag{3}$$

The liner control law that would provide the ideal discrete-time sliding mode, also called the discrete-time equivalent control, is obtained by solving $s(k+1) = 0$ as:

$$u_{eq}(k) = -(\boldsymbol{\sigma}\boldsymbol{\gamma})^{-1}\boldsymbol{\sigma}\mathbf{\Phi}\mathbf{x}(k), \tag{4}$$

assuming $\boldsymbol{\sigma}\boldsymbol{\gamma} \neq 0$. In other words, if the state of the system is initially on the sliding surface $s(k) = 0$, the control (4) will provide the system motion on $s(k) = 0$ and towards the steady state.

III. Minimum Variance Control

The discrete-time model of the plant (1) in *z*-domain is given by:

$$y(k) = \frac{z^{-1}B(z^{-1})}{A(z^{-1})}u(k), \tag{5}$$

where:

$$A(z^{-1}) = z^{-n}\det(z\mathbf{I}-\mathbf{\Phi}), \tag{6}$$

$$B(z^{-1}) = z^{-n+1}\mathbf{d}\,\mathrm{adj}(z\mathbf{I}-\mathbf{\Phi})\boldsymbol{\gamma}, \tag{7}$$

and z^{-1} is a unit delay ($z^{-1} = e^{-sT}$, s is a complex variable).

The switching function is now chosen as:

$$s(k) = C(z^{-1})y(k). \tag{8}$$

The polynomial $C(z^{-1}) = c_0 + c_1 z^{-1} + ... + c_{n-1}z^{-n+1} + c_n z^{-n}$ has all zeros inside the unit disk.

The linear control ensuring $s(k+1) = 0$ is the minimum variance control:

$$u_{mv}(k) = -\frac{F(z^{-1})}{E(z^{-1})B(z^{-1})}y(k). \tag{9}$$

The polynomials $E(z^{-1})$ and $F(z^{-1})$ are the solutions of the so-called Diophantine equation:

$$E(z^{-1})A(z^{-1}) + z^{-1}F(z^{-1}) = C(z^{-1}). \tag{10}$$

It can be easily proved, by implementing Eq. (9) in Eq. (5), taking into account Eq. (10), that $s(k+1) = 0$ is ensured by the control (9), indeed.

IV. Input-output Based Discrete-time Equivalent Control Approach

In order to show that the minimum variance control represents the equivalent control analog for the input-output based VSC systems, we convert the plant model (1) using the following transformation:

$$\mathbf{z}(k) = \mathbf{T}\mathbf{x}(k), \tag{11}$$

where:

$$\mathbf{z}(k) = [\mathbf{z}_1(k) \quad \mathbf{z}_2(k) \quad ... \quad \mathbf{z}_n(k)]^{\mathrm{T}}, \tag{12}$$

$$\mathbf{z}_i(k) = [x_i(k) \quad x_i(k-1) \quad ... \quad x_i(k-n)]^{\mathrm{T}}, \tag{13}$$

and $\mathbf{T}$ is the transformation matrix in the form of:

$$\mathbf{T} = \begin{bmatrix} \mathbf{t} & \mathbf{0} & ... & \mathbf{0} \\ \mathbf{0} & \mathbf{t} & ... & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & ... & \mathbf{t} \end{bmatrix}, \tag{14}$$

$$\mathbf{t} = [1 \quad z^{-1} \quad ... \quad z^{-n}]^{\mathrm{T}}, \tag{15}$$

and we implement the procedure described in Section II for obtaining the discrete-time equivalent control.

The implementation of Eq. (11) in the plant model (1) gives:

$$\mathbf{z}(k+1) = \begin{bmatrix} \mathbf{\Phi_{11}} & \mathbf{\Phi_{12}} & ... & \mathbf{\Phi_{1n}} \\ \mathbf{\Phi_{21}} & \mathbf{\Phi_{22}} & ... & \mathbf{\Phi_{2n}} \\ \vdots & \vdots & & \vdots \\ \mathbf{\Phi_{n1}} & \mathbf{\Phi_{n2}} & ... & \mathbf{\Phi_{nn}} \end{bmatrix}\mathbf{z}(k) + \begin{bmatrix} \boldsymbol{\gamma}_1 \\ \boldsymbol{\gamma}_2 \\ \vdots \\ \boldsymbol{\gamma}_n \end{bmatrix}u(k), \tag{16}$$

with:

$$\mathbf{\Phi_{ij}} = \phi_{ij}\mathbf{I}_{(n+1)\mathrm{x}(n+1)}, \tag{17}$$

$$\boldsymbol{\gamma}_{\mathbf{i}} = \gamma_i \mathbf{t}, \tag{18}$$

since:

$$\mathbf{T\Phi} = \begin{bmatrix} \mathbf{\Phi_{11}} & \mathbf{\Phi_{12}} & ... & \mathbf{\Phi_{1n}} \\ \mathbf{\Phi_{21}} & \mathbf{\Phi_{22}} & ... & \mathbf{\Phi_{2n}} \\ \vdots & \vdots & & \vdots \\ \mathbf{\Phi_{n1}} & \mathbf{\Phi_{n2}} & ... & \mathbf{\Phi_{nn}} \end{bmatrix}\mathbf{T}. \tag{19}$$

The model (16) can be rewritten as:

$$\mathbf{z}_1(k+1) = \mathbf{\Phi_{11}}\mathbf{z}_1(k) + \overline{\mathbf{\Phi}}_{\mathbf{12}}\overline{\mathbf{z}}_2(k) + \boldsymbol{\gamma}_{\mathbf{1}}u(k), \tag{20}$$

$$\overline{\mathbf{z}}_2(k+1) = \overline{\mathbf{\Phi}}_{\mathbf{21}}\mathbf{z}_1(k) + \overline{\mathbf{\Phi}}_{\mathbf{22}}\overline{\mathbf{z}}_2(k) + \overline{\boldsymbol{\gamma}}_{\mathbf{2}}u(k), \tag{21}$$

where:

$$\overline{\mathbf{\Phi}}_{\mathbf{12}} = [\mathbf{\Phi_{12}} \quad \mathbf{\Phi_{13}} \quad ... \quad \mathbf{\Phi_{1n}}], \tag{22}$$

$$\overline{\mathbf{\Phi}}_{\mathbf{21}} = [\mathbf{\Phi_{21}} \quad \mathbf{\Phi_{31}} \quad ... \quad \mathbf{\Phi_{n1}}]^{\mathrm{T}}, \tag{23}$$

$$\overline{\mathbf{\Phi}}_{\mathbf{22}} = \begin{bmatrix} \mathbf{\Phi_{22}} & ... & \mathbf{\Phi_{2n}} \\ \vdots & & \vdots \\ \mathbf{\Phi_{n2}} & ... & \mathbf{\Phi_{nn}} \end{bmatrix}. \tag{24}$$

$$\overline{\boldsymbol{\gamma}}_{\mathbf{2}} = [\boldsymbol{\gamma}_{\mathbf{2}} \quad \boldsymbol{\gamma}_{\mathbf{3}} \quad ... \quad \boldsymbol{\gamma}_{\mathbf{n}}]^{\mathrm{T}} \tag{25}$$

From Eq. (21), $\overline{\mathbf{z}}_2(k)$ is calculated as:

$$\overline{\mathbf{z}}_2(k) = \left(z\mathbf{I}_{(n^2-1)\mathrm{x}(n^2-1)} - \overline{\mathbf{\Phi}}_{\mathbf{22}}\right)^{-1}\overline{\mathbf{\Phi}}_{\mathbf{21}}\mathbf{z}_1(k) + \left(z\mathbf{I}_{(n^2-1)\mathrm{x}(n^2-1)} - \overline{\mathbf{\Phi}}_{\mathbf{22}}\right)^{-1}\overline{\boldsymbol{\gamma}}_{\mathbf{2}}u(k), \tag{26}$$

and substituted in Eq. (20), providing:

$$\mathbf{z}_1(k+1) = \tilde{\mathbf{\Phi}}(z)\mathbf{z}_1(k) + \tilde{\boldsymbol{\gamma}}(z)u(k), \tag{27}$$

with:

$$\tilde{\mathbf{\Phi}}(z) = \mathbf{\Phi_{11}} + \overline{\mathbf{\Phi}}_{\mathbf{12}}\left(z\mathbf{I}_{(n^2-1)\mathrm{x}(n^2-1)} - \overline{\mathbf{\Phi}}_{\mathbf{22}}\right)^{-1}\overline{\mathbf{\Phi}}_{\mathbf{21}}, \tag{28}$$

$$\tilde{\boldsymbol{\gamma}} = \boldsymbol{\gamma}_{\mathbf{1}} + \overline{\mathbf{\Phi}}_{\mathbf{12}}\left(z\mathbf{I}_{(n^2-1)\mathrm{x}(n^2-1)} - \overline{\mathbf{\Phi}}_{\mathbf{22}}\right)^{-1}\overline{\boldsymbol{\gamma}}_{\mathbf{2}}. \tag{29}$$

We choose now the switching function to be:

$$s(k) = \tilde{\mathbf{c}}\mathbf{z}_1(k), \quad \tilde{\mathbf{c}} = [\tilde{c}_0 \quad \tilde{c}_1 \quad ... \quad \tilde{c}_n]. \tag{30}$$

Notice that Eqs. (8) and (30) are equal if the coefficients $c_i = \tilde{c}_i$, $i = \overline{0,n}$. As in Section II, the discrete-time equivalent control is obtained by solving:

$$s(k+1) = \tilde{\mathbf{c}}\tilde{\mathbf{\Phi}}(z)\mathbf{z}_1(k) + \tilde{\mathbf{c}}\tilde{\boldsymbol{\gamma}}(z)u(k) = 0, \tag{31}$$

as:

$$u_{eq}(k) = -(\tilde{\mathbf{c}}\tilde{\boldsymbol{\gamma}}(z))^{-1}\tilde{\mathbf{c}}\tilde{\mathbf{\Phi}}(z)\mathbf{z}_1(k), \tag{32}$$

assuming $(\tilde{\mathbf{c}}\tilde{\boldsymbol{\gamma}}(z))^{-1}$ is realizable. Eqs. (9) and (32) have the similar form, representing the digital filters of the same order. This will be shown by the illustrative example given in the next section.

V. Illustrative Example

To promote the conversion procedure given in the previous section and to show that it gives similar digital filter as the minimum variance control technique, we consider the second order plant model whose transfer function is:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \tag{33}$$

and the continuos-time state-space model:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix} u(t) \quad (34)$$

Under the assumption that a control signal is constant during sampling period, $u(t) = u(kT),\ kT \le t \le (k+1)T$,the discrete-time model in the state-space is obtained from (34) as:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} u(k), \quad (35)$$

using standard discretization procedure, and in *z*-domain:

$$y(k) = \frac{\gamma_1 + (\gamma_2\phi_{12} - \gamma_1\phi_{22})z^{-1}}{1 - (\phi_{11} + \phi_{22})z^{-1} + (\phi_{11}\phi_{22} - \phi_{12}\phi_{21})z^{-2}} u(k-1). \quad (36)$$

For $\zeta = 0.707$, $\omega_n = 10$ and $T = 0.01s$, the models (35) and (36) become:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.9952 & 0.0093 \\ -0.9310 & 0.8636 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.0048 \\ 0.9310 \end{bmatrix} u(k), \quad (37)$$

and

$$y(k) = \frac{0.0048 + 0.0045z^{-1}}{1 - 1.8588z^{-1} + 0.8681z^{-2}} u(k-1). \quad (38)$$

respectively. We also suppose that the switching functions are identical in both cases i.e. $c_0 = \tilde{c}_0$, $c_1 = \tilde{c}_1$ and $c_2 = \tilde{c}_2$ and the polynomial $C(z^{-1})$ is defined as:

$$C(z^{-1}) = (1 - z^{-1} \exp(-2\pi f_{cut} T))^2 \quad (39)$$

with $f_{cut} = 1\text{Hz}$.

The minimum variance control for the plant model (36) is given by:

$$u_{mv}(k) = -\frac{f_0 + f_1 z^{-1}}{g_0 + g_1 z^{-1}} y(k), \quad (40)$$

where:

$$f_0 = c_1 / c_0 + \phi_{11} + \phi_{22} = -0.0194, \quad (41)$$

$$f_1 = c_2 / c_0 - \phi_{11}\phi_{22} + \phi_{12}\phi_{21} = 0.0138, \quad (42)$$

$$g_0 = \gamma_1 = 0.0048, \quad (43)$$

$$g_1 = \gamma_2\phi_{12} - \gamma_1\phi_{22} = 0.0045. \quad (44)$$

The implementation of the procedure elaborated in Section IV yields:

$$u_{eq}(k) = -\frac{\tilde{f}_0 + \tilde{f}_1 z^{-1}}{\tilde{g}_0 + \tilde{g}_1 z^{-1}} y(k), \quad (45)$$

with:

$$\tilde{f}_0 = \phi_{11} = 0.9952, \quad (46)$$

$$\tilde{f}_1 = -\phi_{11}\phi_{22} + \phi_{12}\phi_{21} = -0.8681, \quad (47)$$

$$\tilde{g}_0 = g_0, \quad (48)$$

$$\tilde{g}_1 = g_1. \quad (49)$$

It is obvious from Eqs. (40) and (45), that the minimum variance control approach and the procedure based on the classical equivalent control method with the transformed state-space plant model produce, as a result, two similar digital filters of the same order. Accordingly, we conclude that the minimum variance control can really be treated as the equivalent control analog for the input-output based VSC systems.

As one can see, the control (45) represents the special case of the minimum variance control (40) with:

$$c_2 = 0, \quad (49)$$

$$c_1 = c_0\phi_{22}. \quad (50)$$

It is also evident that the control law from Section IV depends only on plant parameters since the vector $\tilde{\mathbf{c}}$ vanishes during the design procedure. Therefore, the system dynamics in sliding mode can not be chosen freely when the control algorithm (32) is implemented. That is the reason why this approach is not used in the input-output based sliding mode control.

The system responses with nominal plant parameters are shown in Figs. 1 and 2 for both control algorithms: (40) and (45). The minimum variance control law gives better dynamic behavior, which is determined by Eq. (39).

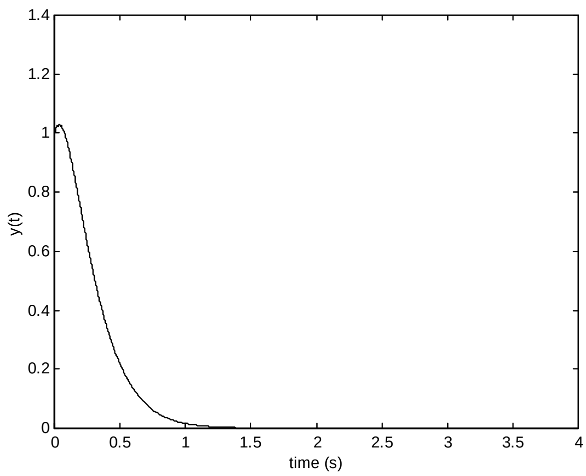

Fig. 1. Response of system with minimum variance control (nominal plant parameters)

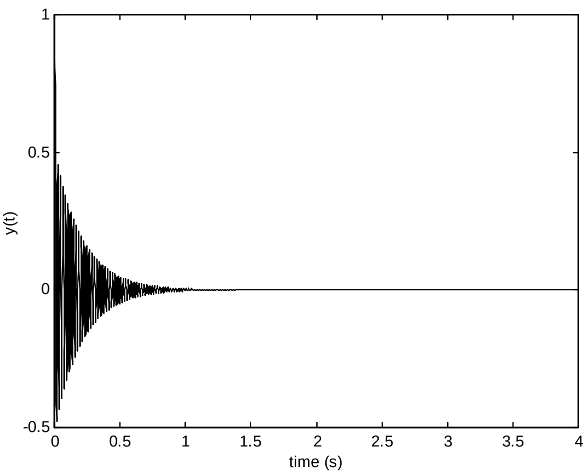

Fig. 2. System response with the control given by Eq. (45) (nominal plant parameters)

The responses of the system with perturbed plant parameters ($\zeta = 1$ and $\omega_n = 12$) are presented in Figs. 3 (Eq. (40)) and 4 (Eq. (45)). The system with the minimum variance control does not change its response significantly, while the control law (45) gives unstable system output, what is expected, since the control law (45) depends only on plant parameters, i.e. there is no control mechanism to compensate the parameter variations.

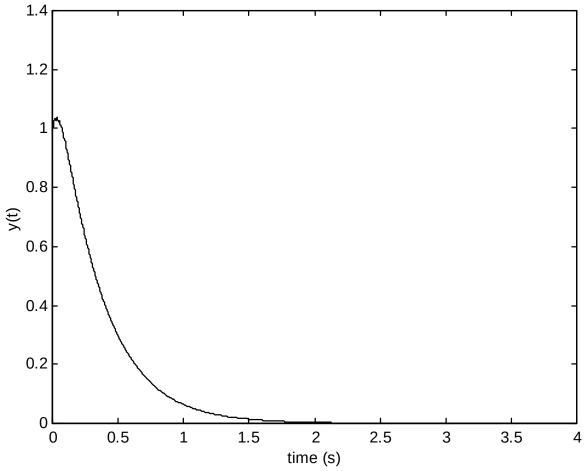

Fig. 3. Response of system with minimum variance control (perturbed plant parameters)

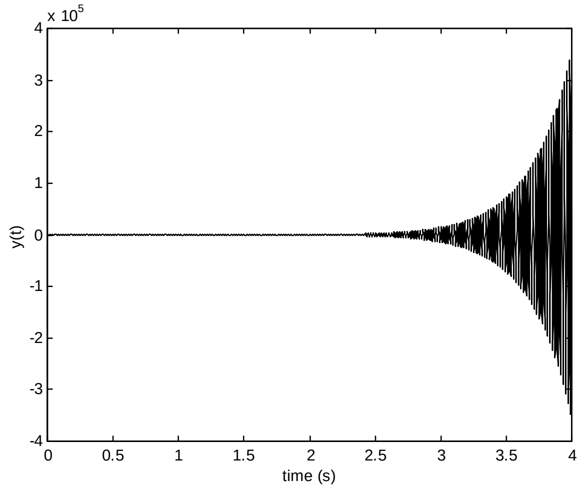

Fig. 4. System response with the control given by Eq. (45) (perturbed plant parameters)

VI. CONCLUSION

The paper establishes the relation between the minimum variance control and the discrete-time sliding mode equivalent control. It is shown that the minimum variance control can be treated as the discrete-time equivalent control analog in the cases when the input-output based variable structure control designs are considered. The proof of this correspondence starts with the transformation of the initial state-space model into the state space model with time-delayed state coordinates. To reduce the obtained model to the model with only plant outputs and its time-delayed values, the *z* transformation is introduced. Then, the standard equivalent control design procedure is implemented, leading to the digital filter of the same order as in the case of the application of the minimum variance control concept. Unfortunately, this approach is limited by a not-freely choice of the system dynamics in sliding mode, since the coefficients of the control algorithm are only dependant on plant parameters. That is the main reason why this approach is not used in the design of input-output based sliding mode control laws.

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