Controlling the Rikitake's nonlinear system with chaotic behaviour by means of synchronization

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Abstract – **In this paper the problems about relay-based control of the nonlinear Rikitake's system are discussed. The control laws are designed on the basis of the necessary condition of the maximum principle. These laws are formed by the state space variables of a second (response) system, which is synchronized with the first (drive) system. The results are confirmed by computer simulation.**

Keywords – **chaos, synchronization, control, conditional Lyapunov exponent.**

I. Introduction

The chaos is a special type of dynamical behaviour of systems, possessing a number of specific features which in essence determine the concept *chaos*. These features are [3, 10]:

- strong sensitivity to tiny variations of the initial conditions and/or the system parameters which means that small differences in the initial conditions lead to substancial differences in the system behaviour;
- the 'motion' of the system in the state space is performed over orbits, which are restricted in a definite area, possessing at the same time a positive Lyapunov exponent;
- these systems have continuous frequency spectrum.

A lot of processes in physics, chemistry, technics, biology, medicine, ecology and economics possess such behaviour [10].

The mathematical model of these systems is a system of nonlinear differential equations of the type:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{p}) , \tag{1.1}$$

where:

- $\mathbf{x} \in \Re^n$ is the state vector;
- $\mathbf{f}(\mathbf{x},\mathbf{p}) = [f_1(\mathbf{x},\mathbf{p}),\ f_2(\mathbf{x},\mathbf{p}),\ \dots,\ f_n(\mathbf{x},\mathbf{p})]^T$
- $\mathbf{p} \in \Re^m$ - is the vector of the variable parameters for which the $m \le n$ condition is satisfied. The change of at least one of the vector elements leads to bifurcations in the system.

It is a characteristic feature of the systems with chaotic behaviour that the linearized system is unstable in all equilibrium points $\mathbf{x}_i^*$, which are the solution of the equation:

$$\mathbf{f}\left(\mathbf{x}^*,\mathbf{p}\right) = \mathbf{0} . \tag{1.2}$$

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1.1. Synchronization of chaotic systems.

The synchronization is a process [1], when two or more connected systems, equivalent or not by structure and parameters, adjust their dynamics to each other. In actual fact this phenomenon is reproducing the state of one of the systems by the other on the basis of information received through a connecting signal.

When dealing with synchronization, two systems S_{dr} (*drive*) and S_{resp} (*response*) are considered:

$$S_{dr}: \quad \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{1.3}$$

and

$$S_{resp}: \quad \dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{f}}(\tilde{\mathbf{x}}), \tag{1.4}$$

with the corresponding solutions $\mathbf{x}(t,t_0,\mathbf{x}(t_0))$ and $\tilde{\mathbf{x}}(t,t_0,\tilde{\mathbf{x}}(t_0))$, where $\mathbf{x} \in \Re^{n_1}$ и $\tilde{\mathbf{x}} \in \Re^{n_2}$ and initial conditions $\mathbf{x}(t_0)$ and $\tilde{\mathbf{x}}(t_0)$.

For $n_1 = n_2$ and $\tilde{\mathbf{f}}(\tilde{\mathbf{x}}) = \mathbf{f}(\mathbf{x})$, the systems S_{resp} and S_{dr} are *identical*.

The solutions $\mathbf{x}(t,t_0,\mathbf{x}(t_0))$ and $\tilde{\mathbf{x}}(t,t_0,\tilde{\mathbf{x}}(t_0))$ of the S_{dr} and S_{resp} systems with initial conditions $\mathbf{x}(t_0)$ and $\tilde{\mathbf{x}}(t_0)$ are *asimptotically synchronized* in relation to a chosen function Q_t, if

$$\lim_{t\to\infty} Q_t\left[\mathbf{x}(t),\tilde{\mathbf{x}}(t)\right] = 0 . \tag{1.5}$$

For identical systems the function most frequently is chosen of the type

$$Q_t = \left\|\mathbf{e}(t)\right\|, \tag{1.6}$$

where

$$\mathbf{e}(t) = \mathbf{x}(t,t_0,\mathbf{x}(t_0)) - \tilde{\mathbf{x}}(t,t_0,\tilde{\mathbf{x}}(t_0)) \tag{1.7}$$

is the difference between S_{dr} and S_{resp}.

Generally the synchronization of chaotic systems is realized by two methods – through one-way and through two-way coupling. In case of one-way coupling the first system is *free* and it *leads* the second, which tracks the dynamics of the first, while in the case of two-way coupling the influence over the two systems is mutual.

In the case of one-way coupling of two autonomous identical systems, which are the subject of the present investigation, most frequently a decomposition of the system to two subsystems (Pecora-Carroll method) is searched [8]:

$$\mathsf{S}_{dr}: \begin{cases} \dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}_1, \mathbf{x}_2), \\ \dot{\mathbf{x}}_2 = \mathbf{f}_2(\mathbf{x}_1, \mathbf{x}_2), \end{cases} \quad (1.8)$$

and

$$\mathsf{S}_{resp}: \begin{cases} \dot{\tilde{\mathbf{x}}}_1 = \mathbf{f}_1(\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2), \\ \dot{\tilde{\mathbf{x}}}_2 = \mathbf{f}_2(\mathbf{x}_1, \tilde{\mathbf{x}}_2), \end{cases} \quad (1.9)$$

where $\mathbf{x}_1 \in \Re^l$, $\mathbf{x}_2 \in \Re^m$, $\tilde{\mathbf{x}}_1 \in \Re^l$, $\tilde{\mathbf{x}}_2 \in \Re^m$, $m + l = n$. The state subvector $\mathbf{x}_1$ of the first subsystem of S_{dr} is applied to the second subsystem of the system S_{resp} for achieving synchronization.

1.2. *Lyapunov exponents*.

A quantitative evaluation of synchronization of chaotic systems is carried out on the basis of the Lyapunov exponents λ_i [1, 3, 8, 9], which are an analogue to the eigenvalues of the linear systems. These exponents are a measure of the convergence or the divergence of the nonlinear systems. A main index for the presence of chaos is the maximum Lyapunov exponent, which is positive. In general case there is no analytical solution for computing the maximum Lyapunov exponent and it is calculated by means of a numerical procedure by the following expression:

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \lim_{|e(t)| \to 0} \frac{|\mathbf{e}(t)|}{|\mathbf{e}(0)|}. \quad (1.10)$$

By the Pecora&Carroll synchronization method

$$\mathbf{e}(t) = \mathbf{x}_2(t) - \tilde{\mathbf{x}}_2(t) \quad (1.11)$$

is the difference between S_{dr} and S_{resp} with initial conditions

$$\|\mathbf{x}_2(0) - \tilde{\mathbf{x}}_2(0)\| \neq 0.$$

The evaluation of the decomposition-type synchronization is carried out by the so called *conditional Lyapunov exponents* (*CLE*), which are calculated by (1.10) with the substitution of the difference (1.11).

II. Control Synthesis

A relay based control of the type:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{p}) + \mathbf{B}u \quad (2.1)$$

has to be determined for the system (1.1).

The Hamilton function for the system (2.1) is:

$$\mathscr{H}(\mathbf{x}, u, t, \boldsymbol{\lambda}) \overset{\Delta}{=} \boldsymbol{\lambda}^{\mathrm{T}}(t)[\mathbf{f}(\mathbf{x}, \mathbf{p}) + \mathbf{B}u] \quad (2.2)$$

and according to the maximum principle the following conditions for the n-dimensional auxiliary vector $\boldsymbol{\lambda}(t) = [\lambda_1(t) \ \cdots \ \lambda_n(t)]^{\mathrm{T}}$ have to be satisfied:

1. For each t (with the exception of the interruption points of $\mathbf{f}$ and u) the Hamiltonian function takes a maximum over the optimal trajectory, i.e.:

$$\frac{\partial \mathscr{H}}{\partial u} = 0. \quad (2.3)$$

2. For each t (with the exception of the interruption points of $\mathbf{f}$ and u) the following condition for the vector $\boldsymbol{\lambda}(t)$ over the optimal trajectory is satisfied:

$$\frac{d\boldsymbol{\lambda}(t)}{dt} = -\frac{\partial \mathscr{H}(\mathbf{x}, u, \boldsymbol{\lambda})}{\partial \mathbf{x}}. \quad (2.4)$$

3. If there is no limitation for the duration of the transient process an additional condition is imposed:

$$\mathscr{H}(\mathbf{x}, u, \boldsymbol{\lambda}) = 0. \quad (2.5)$$

It follows from (2.3) that the control is relay based of the kind:

$$u = \operatorname{sign} s_i(\mathbf{x}), \quad (2.6)$$

where with taking to consideration of (2.4) and the necessary condition for a nonzero vector $\boldsymbol{\lambda}$, the control function over the *i*-th input $s_i(\mathbf{x})$ for third-order systems is obtained from the following equation:

$$\det \begin{bmatrix} f_l(\mathbf{x}) & f_k(\mathbf{x}) \\ -\frac{\partial f_l(\mathbf{x})}{\partial x_i} & -\frac{\partial f_k(\mathbf{x})}{\partial x_i} \end{bmatrix} = 0 \quad \text{for} \quad k \neq i \ \text{ and } \ l \neq i \quad (2.7)$$

and it has the form:

$$s_i(\mathbf{x}) = f_k(\mathbf{x}) \frac{\partial f_l(\mathbf{x})}{\partial x_i} - f_l(\mathbf{x}) \frac{\partial f_k(\mathbf{x})}{\partial x_i} = 0. \quad (2.8)$$

It is accepted that the control function (2.8) is formed from the state variables of S_{resp} and the control is active when the system trajectory enters a given region $\mathscr{L}_i$ in the state space (as shown on fig.1) for which the following condition is satisfied:

$$\mathbf{x}_i^* \in \mathscr{L}_i \in \mathscr{A}. \quad (2.9)$$

$\mathscr{A}$ is the system attractor and the region $\mathscr{L}_i$ is a sphere

$$\mathscr{L}_i = \left\{ \mathbf{x} \in \Re^n : \|\mathbf{x} - \mathbf{x}_i^*\| \leq R \right\} \quad (2.10)$$

around the equilibrium point $\mathbf{x}_i^*$ with a radius R, which has to be such that the condition

$$\mathscr{L}_i \cap \mathscr{A} \neq \varnothing \quad (2.11)$$

to be performed.

This is the *so called* local control [9].

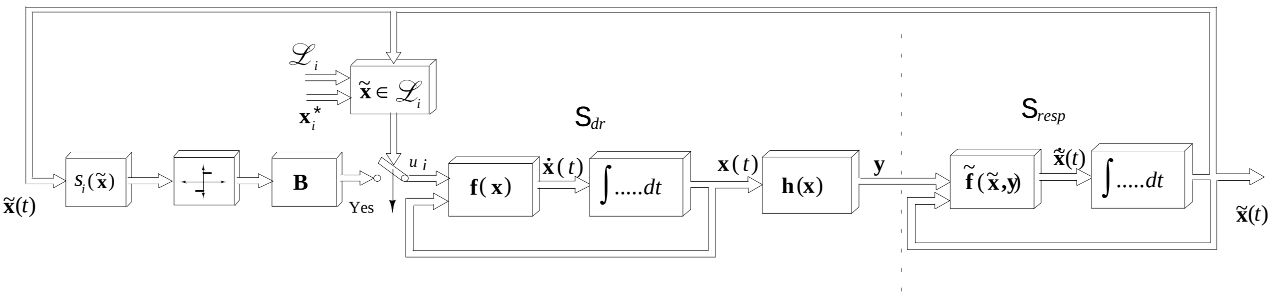

Fig. 1.

III. Rikitake's system

The Rikitake's system [4] represents a model of a double-disk dynamo system and is described by the following equations:

$$\begin{aligned} \dot{x}_1 &= -\mu x_1 + x_2 x_3, \\ \dot{x}_2 &= -\mu x_2 + (x_3 - \mu A)x_1, \\ \dot{x}_3 &= b - x_1 x_2, \end{aligned} \qquad (3.1)$$

where $A = \left(K^2 - K^{-2}\right)$.

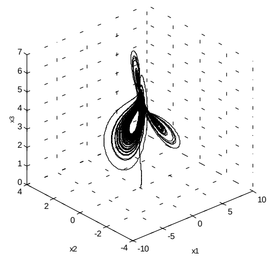

Fig. 2

The system attractor is shown on fig.2. The system has two equilibrium points with the following coordinates:

$$\mathbf{x}_{1,2}^* = \begin{bmatrix} \pm K\sqrt{b} \\ \pm K^{-1}\sqrt{b} \\ \mu K^2 \end{bmatrix}, \quad (3.2)$$

which for parameter values $\mu = 1, \quad K = 2, \quad b = 1$ are

$$\mathbf{x}_{1,2}^* = \begin{bmatrix} \pm 2 \\ \pm 0.5 \\ 4 \end{bmatrix}.$$

The Jacobian of the system (3.1) in the equilibrium points $\mathbf{x}_j^*$ is

$$\mathbf{A}_j = \begin{bmatrix} -\mu & x_{j3}^* & x_{j2}^* \\ x_{j3}^* - \mu A & -\mu & x_{j1}^* \\ -x_{j2}^* & -x_{j1}^* & 0 \end{bmatrix}. \qquad (3.3)$$

Evaluation of the possibility to drive the system into $\mathbf{x}_j^*$ is made by the rank of the matrix

$$\mathbf{Q}_i = \left[\mathbf{B}_i \quad \mathbf{B}_i\mathbf{A}_j \quad \mathbf{B}_i\mathbf{A}_j^2\right]. \qquad (3.4)$$

The analysis shows that

$$\operatorname{rank}\mathbf{Q}_i = 3 \quad \forall\ i \text{ and } j, \qquad (3.5)$$

from which follows that the system is completely controllable in the equilibrium points.

The control functions for the i-th input control by taking into consideration the state variables of S_{resp} are:

$$s_1(\mathbf{x}) = b(x_3 - \mu A) - \mu x_2^2$$
$$s_2(\mathbf{x}) = b x_3 - \mu x_1^2$$
$$s_3(\mathbf{x}) = \mu\left(x_1^2 - A x_1 x_2 - x_2^2\right)$$

The Pecora&Carroll synchronization method is proposed for the Rikitake's system. Table 4.1 shows the three possible decompositions of the system by the given method as well as the calculated according to [12] conditional Lyapunov exponents.

Табл. 4.1.

Controlled system	Connec-ting variable	Conditional Lyapunov exponent
$\dot{\tilde{x}}_2 = -\mu\tilde{x}_2 + (\tilde{x}_3 - \mu A)x_1$ $\dot{\tilde{x}}_3 = b - x_1\tilde{x}_2$	x_1	$\lambda_{1,2} = -0.5$
$\dot{\tilde{x}}_1 = -\mu\tilde{x}_1 + x_2\tilde{x}_3$ $\dot{\tilde{x}}_3 = b - x_1\tilde{x}_2$	x_2	$\lambda_{1,2} = -0.5$
$\dot{\tilde{x}}_1 = -\mu\tilde{x}_1 + \tilde{x}_2 x_3$ $\dot{\tilde{x}}_2 = -\mu\tilde{x}_2 + (x_3 - \mu A)\tilde{x}_1$	x_3	$\lambda_1 = -4.76$ $\lambda_2 = 2.76$

Synchronization of the *response* system with the *drive* system is only possible with x_1 and x_2 as driving variables since one of the conditional Lyapunov exponents in the case of x_3 as driving signal is positive.

III. Experimental results

Synchronization with x_1 and x_2 as driving variables is done. A control function is then applied to the first, to the second and to the third input of the original system using the state variables of the synchronized response system. The following figures show the errors (the differences) by synchronization with x_1 and x_2 and control over the three inputs.

- First-input control, synchronization by x_1.

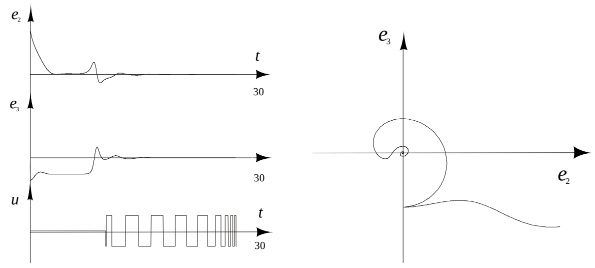

Fig. 3

- First-input control, synchronization by x_2 .

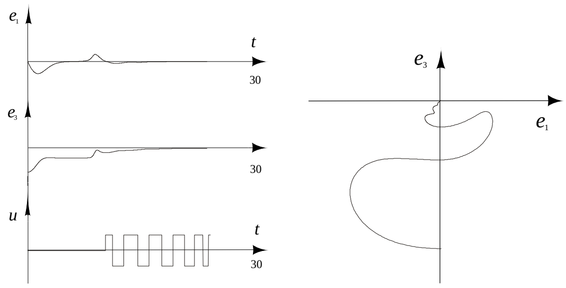

Fig. 4

- Second-input control, synchronization by x_1

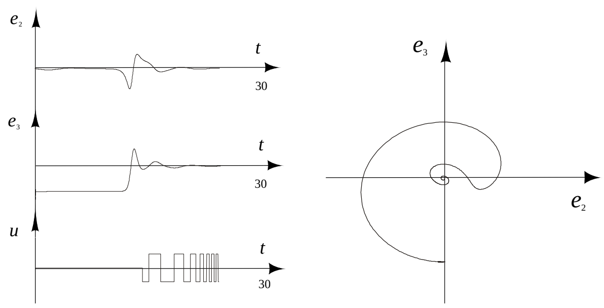

Fig. 5

- Second-input control, synchronization by x_2

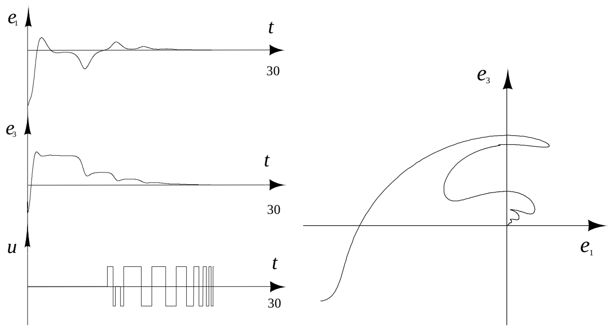

Fig. 6

- Third-input control, synchronization by x_1

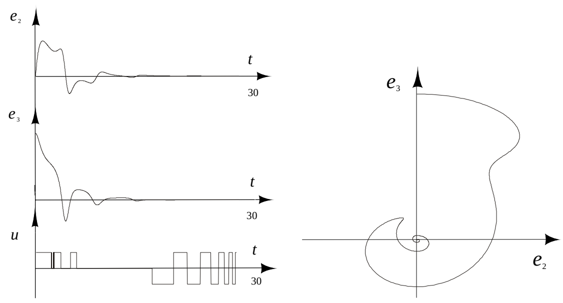

Fig. 7

- Third-input control, synchronization by x_2

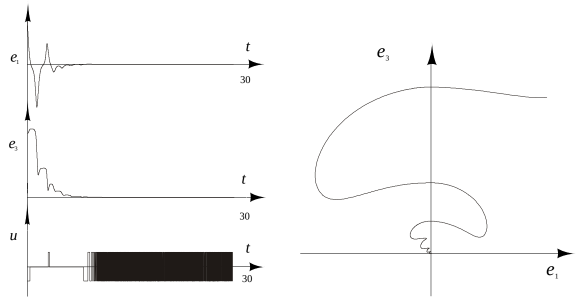

Fig. 8

III. Conclusion

In this paper control through synchronization for the Rikitake's system is proposed. The investigations made show that this control in essence little differs from the control through direct use of the state vector.

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