Sampling factors and amplitude errors during the sinusoidal and cosinusoidal signal conversion

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Abstract**: It is discussed a new approach to amplitude errors calculation during the analog to digital signal conversion due to non sampling the analog signal into its maximal value and in particular during the sinusoidal and co-sinusoidal signal conversion. Formulas for calculating the sampling factor, the sampling frequency Fd and the converter bits n are given.**

Keywords**: amplitude errors, sinusoidal and cosinusoidal signal conversion, sampling factor N**

I. Introduction

The conversion of the analog signal into digital codes and the conversion of the digital codes into analog signals (or more precisely conversion of the analog signal into analog staircase function) are two of the most important problems during the digital signal processing. In order to resolve these problems a number of parameters should be calculated. In this paper we will concentrate on calculating the sampling factor N, sampling frequency Fd and number of converters bits n in order to obtain a given accuracy or amplitude error.

II. Sampling factor N=Fd/Fs

We will introduce a parameter called "sampling factor" ("encoding factor" or "dicsretization factor") with the formulas:

1. For sinusoidal signal (SS) or cosinusoidal signal (CS)

$$N=Fd/Fs>0 \text{ or} \qquad (1)$$

2. For band wide signal

$$N=Fd/Fmax>0 \qquad (2)$$

Where

Fd is the discretization (sampling, encoding) frequency

Fs is the signal frequency (in this case the frequency of the SS or CS)

Fmax is the maximum frequency of interest in the signal band

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In the present paper we will deal with sampling factor N=Fd/Fs>=2 because the evaluation of the errors and the reconstruction of the sampled signal is simpler that in the case with N<2.

III. Basic test analog signals and theirs parameters

The basic analog test signals in the electronics are: direct current (DC), SS, CS, triangular, trapezoidal and rectangular (usually with Thighlevel=Tlow level=0.5*Tperiod) signals.

We will discuss the amplitude error during the SS and CS signals conversion into digital codes with ADC, because with acceptable from the practical point of view accuracy most of the analog signals could be considered as an algebraic sum of DC, SS and CS signals. Moreover, the SS and CS are the basic alternative current (AC) test signals.

As it is well known the SS and CS with DC component are characterized with five basic "parameters": amplitude Am, function (sinus or cosinus), frequency (F or ω), phase (φ) and direct current component B and are given with the followings equations:

$$A = Am * \sin(2*\pi*F*t+\varphi)+B \qquad (3)$$

$$A = Am * \cos(2*\pi*F*t+\varphi)+B \qquad (4)$$

There are two main approaches for analog signal reconstruction from its digital samples:

- Mathematical approach. The parameters of the analog signal are calculated using mathematical methods (models) from the digital samples. Additional points (samples) could be calculated and then the signal is reconstructed. In order to simplify the problem we will admit that the function, represented with the signal (sinus or cosinus in particular) is known. In this case in order to find the four basic parameters Am, F, φ and B a system with four unknowns and four equations should be resolved and a set of four samples A1, A2, A3 and A4 during one period should be used:

$$A1 = Am * \sin(2*\pi*F*t+\varphi)+B \qquad (5)$$

$$A2 = Am * \sin(2*\pi*F*t+\varphi)+B \qquad (6)$$

$$A3 = Am * \sin(2*\pi*F*t+\varphi)+B \qquad (6)$$

$$A4 = Am * \sin(2*\pi*F*t+\varphi)+B \qquad (7)$$

- Direct approach. During the direct signal reconstruction the digital samples (codes) are send directly to the DAC and the produced signal (the analog staircase function) or the "copy" is compared with the "original" analog signal at the ADC input.

In this paper we will use the second approach because it is simple, useful and easy for implementation in electronic equipment.

If the function of the signal (for example sinus or cosinus) is known only four basic "parameters" of the signal will be converted into digital codes and it could be estimated that at least four samples during one period are needed for direct reconstruction of SS or CS (Table 1).

Table 1.
Supposition about the minimum number of samples for arithmetic calculation of "parameters" of the sinusoidal and cosinusoidal signals

"Unknown parameters"	Formula for SS	Nmin
Amplitude, frequency, phase, direct current component B	A = Am * sin (2*π*F*t+φ)+B	4
Amplitude, frequency, phase	A = Am * sin (2*π*F*t+φ)	3
Amplitude, frequency	A = Am * sin (2*π*F*t)	2
Amplitude	A = Am * const	1

Notes: 1. (*) With enough amplitude of the samples.
2. Nmin =Fdmin/Fs= supposed minimal number of samples per period to reconstruct directly signal "parameters".

It should be noted that the staircase approximation of the analog signal during the analog to digital conversion depends on three basic factors: 1/ the sampling factor N=Fd/Fs, 2/ the number of the converters bits and 3/ the angle of the first sample φ0.

Moreover, the analog to digital converter could be called "Analog to Staircase Function Converter/Approximater" (ASFC or ASFA) and the block built by an analog to digital converter and a digital converter could be called "Analog to Analog Staircase Function Converter" (AASFC).

IV. Calculating the maximum amplitude error during the SS and CS conversion

When we know the sampling factor N we could calculate the maximum possible amplitude error Emax, and when we have the maximum allowed amplitude error Emax during the signal conversion we could calculate the sampling factor N=Fd/Fs. This "bi-directional" approach is illustrated with the formulas illustrated in the Table 2.

Table 2.
The "bi-directional" relation between the sampling factor N=Fd/Fs and the maximum amplitude error Emax when sampling sinusoidal and cosinusoidal signal.

Analog Signal	Sampling factor N=Fd/Fs
SS	N= 180(90-arcsin(1-Emax))>=2
CS	N= 180/arccos (1-Emax) >=2
	Amplitude Error Eamax
SS	1>=Emax = (1-sin(90-(180/N))>=0.
CS	1>=Emax = (1-cos (180/N))>=0.

Note: The formulas are valid for SS and CS with N>=2 and with an ideal converter. When N<2 a different approaches should be used.

Once the sampling factor N is calculated it is possible to calculate the sampling frequency Fd with the formula

$$Fd = N*Fmax = N*Fs \qquad (8)$$

The utility of the formulas given in Table 2 could be seen in Table 3 and Table 4.

Table 3.
The relation between the maximum amplitude error Emax and the sampling factor N=Fd/Fs from 2 to 5 with step 1 during the SS or CS conversion with ideal ADC (φd/2=αmax)

N	φd	A(αmax)	A(αmax) %	Emax[%]
2.0(!)	180	0	0	100 (!)
2.1	171.4	0.0747	7.47	92.5
2.2	163.6	0.142	14.2	85.8
2.3	156.5	0.203	20.3	79.1
2.4	150	0.259	25.9	74.1
2.5	144	0.309	30.9	69.1
2.6	138.5	0.355	35.5	64.5
2.7	133.3	0.396	39.6	60.4
2.8	128.6	0.434	43.4	56.6
2.9	124.1	0.468	46.8	53.2
3.0	120	0.50	50	50
3.1	116.1	0.529	52.9	47.1
3.2	112.5	0.556	55.6	44.4
3.3	109.1	0.581	58.1	42
3.4	105.9	0.603	60.3	39.7
3.5	102.9	0.623	62.3	37.7
3.6	100	0.643	64.3	35.7
3.7	97.2	0.661	66.1	33.9
3.8	94.7	0.677	67.7	32.3
3.9	92.3	0.693	69.3	30.7
4.0 (!)	90	0.707	70.7	29.3 (!)
4.1	87.8	0.721	72.1	27.9
4.2	85.7	0.733	73.3	26.7
4.3	83.7	0.745	74.5	25.5
4.4	81.8	0.756	75.6	24.4
4.5	80	0.766	76.6	23.4
4.6	78.3	0.776	77.6	22.4
4.7	76.6	0.785	78.5	21.5
4.8	75	0.793	79.3	20.7
4.9	73.5	0.801	80.1	19.9
5.0	72.0	0.809	80.9	19.1

Notes: 1/ When N=2 Emax could be from 0% to -100%.
2/ When N=4, Emax could be from 0% to –29.3%.

According the Table 2 it easy to calculate the sampling factor N when the maximum amplitude error Emax is known and vice versa. In Table 2 and Table 3 the following abbreviations are used:

- N =Fd/Fs is the sampling factor
- φd=360/N is the sampling angle (the angle between two successive samples)

- φd/2=αmax =360/(2*N) is the angle of the maximum deviation from the maximal value of the SS or CS (the angle of the maximal amplitude error Emax)
- A(αmax) – is the maximum value of the guaranteed digital code (sample) when a converter with infinity number of bits (n→ ∞) and with the conversion error equal to zero (Eadc=0 or Edac=0) is used.
- A(αmax) % is A(αmax) in percentage of the maximum value (100%) of SS or CS
- Emax[%] = 100% - A(αmax)% is the maximum amplitude error or the maximum deviation from the maximal value of the sampled SS or CS

Table 4.
The relation between Emax and N=Fd/Fs with step 2 to 39 with step 1.

N from 2 to 19			**N from 20 to 39**	
N	Emax [%]		N	Emax [%]
2	100		21	1.11
3	50		22	1.02
4	29.3		23	0.9314
5	19.1		24	0.856
6	13.4		25	0.789
7	9.9		26	0.729
8	7.61		27	0.676
9	6.09		28	0.629
10	4.89		29	0.586
11	4.05		30	0.548
12	3.40		31	0.513
13	2.91		32	0.482
14	2.51		33	0.453
15	2.18		34	0.427
16	1.92		35	0.403
17	1.70		36	0.381
18	1.51		37	0.360
19	1.36		38	0.342
20	1.23		39	0.324

V. Calculating number of converters bits

Is was found that when the sampling factor N for SS or CS is know and 2<N<64 we could calculate approximately the number of converters bits in order the converter to be considered as an "ideal converter" with the following formula:

$$n1>=lg(N)+2=lg(Fd/Fs)+2, [bit] \quad (9)$$

where lg is a logarithm in base 2.

When the amplitude error Emax (0<Emax<1) is known a more general formula given below could be used to calculate the minimum number of converter's bits in order the converter to be considered as an "ideal converter"

$$n2>=lg(1/Emax)+2, [bit] \quad (10)$$

When signal to noise ratio (SNR) is known in decibels the well know formula for calculating the number of converters bits could be used

$$n3>= (SNR-1.76)/6.02, [dB] \quad (11)$$

When we could apply all of the three approaches mention before we should choose the maximum possible value of n, according to the formula:

$$n>=max(n1,n2,n3) \quad (13)$$

VII. A general approach for calculating the sampling factor N

Because SS and CS have five basic "parameters", we could say that five basic errors are introduced during the signal conversion:

- amplitude error Ea,
- function error Efn,
- frequency error Efr,
- phase error Eφ,
- direct current error Edc (Normally, Edc=0 when N=Fd/Fs=4*k, k=1, 2, 3, ...).

In order to calculate the total conversion factor N=Fd/Fs we could apply the following approach:

- We are evaluating the five basic errors: amplitude errors Ea, frequency error Efr, function error Efn, phase error Eφ and the direct current error Edc for the purposes of the application.
- We are calculating the five sampling (conversion) factors: amplitude sampling factor Na, function sampling factor Nfh, frequency sampling factor Nfn, phase sampling factor Nφ and direct current error sampling factor Ndc according to the corresponding errors.
- We are calculating the total sampling factor N total as a maximal value of the five sampling factor calculated before

  $$Ntotal= max(Na, Nfn, Nfr, N\varphi, Ndc) \quad (14)$$

- We are calculating the sampling frequency Fd according to the formula Fd=Ntotal*Fs.

The method discussed above is guaranteeing that the conversion error for each of the parameter is less than the limiting values (Ea, Efn, Efr, Edc and Eφ).

VIII. Conclusion

The paper is giving simple and practical approach for calculating during the SS and CS conversion several important parameters:

- the sampling factor N=Fd/Fs;
- the maximal amplitude error Emax due to non sampling the SS/CS into its maximal value;
- the sampling frequency Fd;
- the minimal number of converters bits n, in order to consider the converter as "an ideal converter" (in order to neglect the converter's error and to keep only the error from the sampling factor N=Fd/Fs).

The equations given in Table 2 are useful for evaluating the maximum amplitude error Emax when sampling a SS and CS analog signal which is not possible with the Theorem of Shannon-Kotelnikov-Whittaker.

Moreover, we have found that sampling factor N=Fd/Fs=4 is guaranteeing a maximum difference between the amplitude of the "original" SS or CS and corresponding maximal digital code (the produced approximated staircase function by the ADC) less than –29.3% or –3dB.

It is important to note that the maximal amplitude error E(N,n) and the root-mean-square error E(N,n)rms during the analog to digital (or staircase function) depends on two principal factors:

- the error E(N) from the sampling factor N=Fd/Fs;
- the error E(n) from the finite number of bits into the digital word n.

We could give the following two formulas:

$$E(N.n)max = E(N)+E(n) \qquad (15)$$

and

$$E(N.n)rms = sqr(E(N)*E(N)+E(n)*E(n)) \qquad (16)$$

The method and equipment for sampling and direct signal reconstruction is described in [1]. Several practical circuits with ADC and DAC are published in [2]. The theorem of Shannon-Kotelnikov-Whittaker may be found in many sources, e.g. [3 and 4].

IX. References

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