Control of Chaotic System by Combined Synchronization

Radoslav H. Radev[1], Dragomir P. Chantov[2]

Abstract – **In this paper synthesis of a relay-based control of chaotic systems is proposed for the cases when only one of the state variables is accessible. A second identical chaotic system is constructed and the two systems are synchronized on the basis of the accessible variable. A combined approach is proposed to achieve synchronization, which combines the advantages of two of the known synchronization methods. Only variables of the second system take part in the control.**

Keywords – **chaotic systems, chaotic synchronization, control, Pontryagin's maximum principle.**

I. Introduction

The chaotic systems are nonlinear continuous or discrete systems which possess complex dynamical behaviour by certain conditions and this behaviour is characterized by a strange attractor in the state space, a positive Lyapunov exponent and a specific type of Poincare section. The main feature of these systems is their extreme sensitivity of the initial conditions. Due to their character, the chaotic systems are for long time considered as non-controllable systems. Ott, Grebogi и Yorke [1] however suggest a method for their control and since then the proposed methods for control and stabilization of chaos increase continuously. On the other hand in the last years there is a growing interest to another phenomenon from the field of the chaotic dynamics - this is the so called chaotic synchronization. The aim here is to connect two chaotic systems in such a way that the dynamics of one of the systems to be dependent of the dynamics of the other. As the control as well as the synchronization of chaotic systems find application in different fields of the technics [2,3].

In this paper a specific task is considered, which involves a combination of the problems about chaotic control and synchronization. The main idea is to construct control functions which will stabilize a given chaotic system on the basis of the Pontryagin's maximum principle. A second (auxiliary in this case) identical chaotic system is built, which is synchronized with the main one by a connecting function in which only one of the state variables takes part. After identical synchronization between the two systems is achieved, the control function, in which only variables from the auxiliary system take part, is applied to the first system. This control function stabilizes the main chaotic system into a preliminarily selected unstable fixed point. To exemplify the proposed control approach we choose the Willamowski-Rossler chaotic system, which describes the processes in a chemical reactor.

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II. Control and Synchronization of Chaotic Systems

Control In the general case a continuous chaotic system can be described by the following nonlinear equation:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, \mathbf{p}) , \tag{1}$$

where $\mathbf{x} \in \Re^n$ is the state variables vector of the system, $\mathbf{p} \in \Re^k$, $k<n$ is the parameters vector and $\mathbf{f}$ is a nonlinear function.

There exist different methods to control the chaotic systems. In this paper control synthesis, based on the Pontryagin's maximum principle, is proposed. On the basis of the necessary conditions of the maximum principle [3] we will seek an external force control of the type:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}) + \mathbf{B}\mathbf{u} , \tag{2}$$

where $\mathbf{B}$ is a column-vector of the type:

$$\mathbf{B} = \begin{bmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix}$$

We will say the control is an *i-th input control*, if $b_i \neq 0$ and $b_j = 0$ for $\forall j \neq i$. An auxiliary vector

$$\boldsymbol{\lambda}(t) = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}$$

is introduced for the formulation of the maximum principle. Then the so called Hamiltonian function is composed:

$$H(\mathbf{x}, \mathbf{u}, t, \boldsymbol{\lambda}) = \boldsymbol{\lambda}^T(t)\mathbf{f}(\mathbf{x}, \mathbf{u}, t) \tag{3}$$

From the maximum principle [3] the Hamiltonian function has a maximum over the optimal system trajectory, i.e.:

$$\frac{\partial H}{\partial \mathbf{u}} = 0 , \tag{4}$$

from which for λ_i we obtain

$$\lambda_i = 0 .$$

(For each t with exception of the interruption points of **f** and **u** the following condition is fulfilled over the optimal trajectory:

$$\frac{d\boldsymbol{\lambda}(t)}{dt} = -\frac{\partial H}{\partial \mathbf{x}} = 0 \qquad (5)$$

If the terminal time is not given, the additional condition:

$$H(\mathbf{x},\mathbf{u},\boldsymbol{\lambda}) = 0 \qquad (6)$$

is imposed.

It follows from (4) that the control will be relay-based from the type:

$$u = k\, sign\, s_i(\mathbf{x}), \qquad (7)$$

where s_i are the control functions, which satisfy the condition for a non-zero vector $\boldsymbol{\lambda}$.

By taking into consideration (4), (5) and (6) for third-order systems and *i*-th input control only one variant of control function s_i for each i is possible. It is obtained from the following system:

$$\det\begin{bmatrix} f_l(\mathbf{x}) & f_k(\mathbf{x}) \\ -\frac{\partial f_l(\mathbf{x})}{\partial x_i} & -\frac{\partial f_k(\mathbf{x})}{\partial x_i} \end{bmatrix} = 0 \quad \text{for } k \neq i \text{ and } l \neq i. \qquad (8)$$

From (8) the control function is:

$$s_i(\mathbf{x}) = f_k(\mathbf{x})\frac{\partial f_l(\mathbf{x})}{\partial x_i} - f_l(\mathbf{x})\frac{\partial f_k(\mathbf{x})}{\partial x_i} = 0. \qquad (9)$$

Synchronization. In the general case when speaking of chaotic synchronization we take two connected chaotic systems of the type:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad (10)$$

$$\dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{f}}(\tilde{\mathbf{x}}), \qquad (11)$$

with the corresponding solutions $\mathbf{x}(t,t_0,\mathbf{x}(t_0))$ and $\tilde{\mathbf{x}}(t,t_0,\tilde{\mathbf{x}}(t_0))$, where $\mathbf{x} \in \Re^{n_1}$, $\tilde{\mathbf{x}} \in \Re^{n_2}$, and the initial conditions of the two systems are $\mathbf{x}(t_0)$ and $\tilde{\mathbf{x}}(t_0)$. For $n_1 = n_2$ and $\tilde{\mathbf{f}}(\tilde{\mathbf{x}}) = \mathbf{f}(\mathbf{x})$ the two systems are identical. The solutions $\mathbf{x}(t,t_0,\mathbf{x}(t_0))$ and $\tilde{\mathbf{x}}(t,t_0,\tilde{\mathbf{x}}(t_0))$ of the systems (10) and (11) with initial conditions $\mathbf{x}(t_0)$ and $\tilde{\mathbf{x}}(t_0)$ are *identically synchronized* [2,4] if the following function

$$Q_t = Q_t[\mathbf{x}(t),\tilde{\mathbf{x}}(t)] = \|\mathbf{e}(t)\| \equiv 0 \qquad \forall t > 0, \qquad (12)$$

where $\mathbf{e}(t)$ is the difference function between the two systems:

$$\mathbf{e}(t) = \mathbf{x}(t,t_0,\mathbf{x}(t_0)) - \tilde{\mathbf{x}}(t,t_0,\tilde{\mathbf{x}}(t_0)). \qquad (13)$$

The equality (12) means that after starting the two systems from different initial conditions, some time later they will begin to oscillate identically in the generalized state space. However this will only be possible if the synchronization process is stable. Measure of the stability of the synchronization give the *conditional Lyapunov exponents*. It is also accepted to refer to the system (10) as Master, and to system (11) as Slave.

The two main approaches for chaotic synchronization are the decomposition methods and the feedback ones. The common between them is that they define some type of connection between the two systems. By the decomposition methods one mentally "decomposes" the Master system into two or more parts, one of which drives the Slave system by direct substitution of some of its variables with the variables of the driving part of the decomposed Master system. Generally the two systems are described by the following equations [2,4]:

$$\textbf{Master} \quad \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{h}(\mathbf{x})), \qquad (14)$$

$$\textbf{Slave} \quad \dot{\tilde{\mathbf{x}}} = \mathbf{f}(\tilde{\mathbf{x}},\mathbf{h}(\mathbf{x})), \qquad (15)$$

where $\mathbf{h}(\mathbf{x})$ is the driving part of the decomposed Master which drives the Slave system.

In the case of the feedback methods with one-way coupling a signal, proportional to the difference between the two systems, is added to the Slave system [4]:

$$\textbf{Master} \quad \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad (16)$$

$$\textbf{Slave} \quad \dot{\tilde{\mathbf{x}}} = \mathbf{f}(\tilde{\mathbf{x}}) + \alpha\mathbf{E}(\mathbf{x}-\tilde{\mathbf{x}}), \qquad (17)$$

where α is the feedback gain and **E** is the coupling matrix with a proper dimension.

In the present paper we suggest a concurrent application of the two approaches [5]:

$$\textbf{Master} \quad \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},x_i), \qquad (18)$$

$$\textbf{Slave} \quad \dot{\tilde{\mathbf{x}}} = \mathbf{f}(\tilde{\mathbf{x}},x_i) + \alpha\mathbf{E}(\mathbf{x}-\tilde{\mathbf{x}}), \qquad (19)$$

where the decomposition part of the coupling is restricted to the *partial replacement* method, where the connection is only by a single variable x_i, substituted only in one place in the Slave system. The second part of the connection (the feedback coupling) can be selected in such way that only the same variable x_i to take part in it and the whole connection between the Master and the Slave system to be only with one variable.

Control by synchronization. The idea about control of chaotic systems by means of a synchronized system is shown on fig.1.

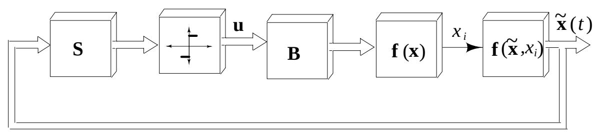

Fig.1. Control by synchronization

If we suppose that only one variable x_i of the Master system is accessible and the aim is to stabilize this system, the control functions (9), in which in the general case all state variables of the Master system take part, can not be obtained. In such a case we can construct a Slave system and synchronize the two systems by the accessible variable x_i. Then the proposed by us combined approach for synchronization (18) and (19) has the advantage that it permits many different variants of connection, even with only one variable. From all these variants we can choose that with the shortest transient process and to control the Master system again with control of type (7), but now the control function (9) is:

$$s_i(\tilde{\mathbf{x}}) = f_k(\tilde{\mathbf{x}})\frac{\partial f_l(\tilde{\mathbf{x}})}{\partial \tilde{x}_i} - f_l(\tilde{\mathbf{x}})\frac{\partial f_k(\tilde{\mathbf{x}})}{\partial \tilde{x}_i} = 0. \qquad (20)$$

The proposed approach for control by synchronization will be illustrated with a concrete third-order chaotic system, since most of the known chaotic systems are relatively simple third-odred nonlinear continuous systems.

III. Willamowski-Rossler System

The Willamowski-Rossler (WR) [6, 7] describes the processes in a chemical reactor and is given with the following equations:

$$\begin{aligned} \dot{x}_1 &= f_1(\mathbf{x}) = k_1x_1 - k_{-1}x_1^2 - k_2x_1x_2 + k_{-2}x_2^2 - k_4x_1x_3 + k_{-4}, \\ \dot{x}_2 &= f_2(\mathbf{x}) = k_2x_1x_2 - k_{-2}x_2^2 - k_3x_2 + k_{-3}, \\ \dot{x}_3 &= f_3(\mathbf{x}) = -k_4x_1x_3 + k_{-4} + k_5x_3 - k_{-5}x_3^2, \end{aligned} \tag{21}$$

where the variables x_i are the concentrations of the species in the reactor and they can take only positive values. The system has 10 parameters $k_{\pm i}$, with nominal values, by which chaos is present in the system:

$k_1 = 31.2$, $k_{-1} = 0.2$, $k_2 = 1.572$, $k_{-2} = 0.1$, $k_3 = 10.8$, $k_{-3} = 0.12$, $k_4 = 1.02$, $k_{-4} = 0.01$, $k_5 = 16.5$ i $k_{-5} = 0.5$

For these values the system evolves chaotically, which is evident by the presence of a typical chaotic attractor in the system state space, which is shown on fig.2.

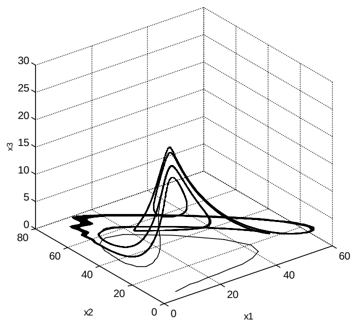

Fig.2. Chaotic attractor of the WR system

When analizing a chaotic system it is important to find its equilibrium or fixed points, i.e. the points for which:

$$\mathbf{f}(\mathbf{x}) = 0 . \tag{22}$$

By means of the program Matlab the fixed points of the WR system are calculated for the nominal values of the system variables and we find that the system has eight fixed points (27):

$\mathbf{x}_1^* = [84.21 \quad 1216 \quad -139]^T$, $\mathbf{x}_2^* = [8.31 \quad 22.7 \quad -1e-3]^T$,

$\mathbf{x}_3^* = [-3e-4 \quad 0.01 \quad -6e-4]^T$, $\mathbf{x}_4^* = [-701 \quad -11130 \quad -1e-5]^T$,

$\mathbf{x}_5^* = [156 \quad -5e-4 \quad 7e-5]^T$, $\mathbf{x}_6^* = [7.354 \quad 7.354 \quad 17.99]^T$,

$\mathbf{x}_7^* = [1.32 \quad 0.014 \quad 30.32]^T$, $\mathbf{x}_8^* = [0.004 \quad 0.01 \quad 32.99]^T$.

Only the points $\mathbf{x}_6^* \div \mathbf{x}_8^*$ have real meaning (there aren't negative concentrations x_i). The point $\mathbf{x}_8^*$ is a stable fixed point (the eigenvalues of the $[p\mathbf{I} - \mathbf{J}_8]$ matrix are all negative, $\mathbf{J_8}$ is the system Jacobian in this point) and hence this point is not of interest for the control, where the purpose is to stabilize the system into an unstable fixed point. the point $\mathbf{x}_7^*$ due to its proximity to $\mathbf{x}_8^*$ is also not of interest notwithstanding the fact that it is unstable. Then we set the aim of the control to stabilize the system into the unstable fixed point $\mathbf{x}_6^*$. We can evaluate the controllability of the system around this point from the matrix:

$$\mathbf{Q}_{6i} = \begin{bmatrix}\mathbf{B}_i & \mathbf{J}_6\mathbf{B}_i & \mathbf{J}_6^2\mathbf{B}_i\end{bmatrix}, \tag{23}$$

where $\mathbf{J_6}$ is the system Jacobian in the point $\mathbf{x}_6^*$. For each *i* the matrix is of full rank, i.e. the system is controllable around the point $\mathbf{x}_6^*$.

IV. Synchronization and Control by Synchronization of the WR System

A Slave system according to (19) is synthesized. For the synchronization we choose a combination of the methods of the partial replacement, by which one variable x_i from the Master system substitutes its corresponding variable of the Slave system only in one place; and the standard one-way coupling, by which there are three variants for the **E** matrix:

$$1.\mathbf{E} = \begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix};\ 2.\mathbf{E} = \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix};\ 3.\mathbf{E} = \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}. \tag{24}$$

With these limitations imposed, there exist 45 different variants of connection between the two systems, which are examined in detail in [5]. 12 of them are with a connecting signal, in which only one state variable is present and the synchronization scheme is stable (the calculated conditional Lyapunov exponents [2] are all negative). To illustrate the proposed in part II control approach we will examine one variant in detail. By this variant the Master system is given by (21) and the slave system is:

$$\begin{aligned} \dot{\tilde{x}}_1 &= k_1\underline{x_1} - k_{-1}\tilde{x}_1^2 - k_2\tilde{x}_1\tilde{x}_2 + k_{-2}\tilde{x}_2^2 - k_4\tilde{x}_1\tilde{x}_3 + k_{-4} + \underline{\alpha(x_1 - \tilde{x}_1)}, \\ \dot{\tilde{x}}_2 &= k_2\tilde{x}_1\tilde{x}_2 - k_{-2}\tilde{x}_2^2 - k_3\tilde{x}_2 + k_{-3}, \\ \dot{\tilde{x}}_3 &= -k_4\tilde{x}_1\tilde{x}_3 + k_{-4} + k_5\tilde{x}_3 - k_{-5}\tilde{x}_3^2. \end{aligned} \tag{25}$$

For $\alpha = 10$ the calculated conditional Lyapunov exponents are $\lambda_{1,2} = -4.73$, $\lambda_3 = -47.07$, i.e. stable synchronization between the two systems will exist. This is shown on fig.3, which depicts the differences $e_i(t) = x_i(t) - \tilde{x}_i(t)$. The initial conditions of the systems are $\mathbf{x}(0) = [5 \quad 2 \quad 1]^T$ and $\tilde{\mathbf{x}}(0) = [6 \quad 1 \quad 0]^T$.

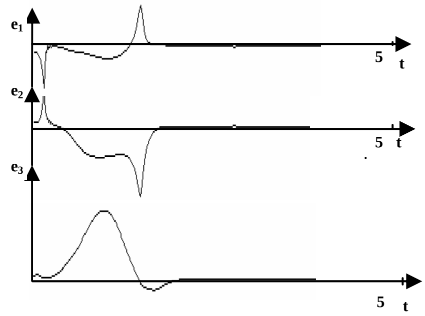

Fig.3. Differences $e_i(t) = x_i(t) - \tilde{x}_i(t)$

After identical synchronization between the two systems is achieved, we can use the Slave system to stabilize the Master system according to fig.1. The control functions for first-, second- and third-input control are calculated according to (20) with the Slave system variables:

$$s_1(\tilde{\mathbf{x}}) = a\tilde{x}_2 + b\tilde{x}_3 + (c - d\tilde{x}_3 - f\tilde{x}_2)\tilde{x}_2\tilde{x}_3\,, \qquad (26)$$

$$a = k_2k_{-4},\; b = k_{-3}k_4,\; c = (k_2k_5 - k_3k_4),\; d = k_2k_{-5},\; f = k_{-2}k_4\,,$$

$$s_2(\tilde{\mathbf{x}}) = (-k_2\tilde{x}_1 + 2k_{-2}\tilde{x}_2)\,f_3(\tilde{\mathbf{x}})\,, \qquad (27)$$

$$s_3(\tilde{\mathbf{x}}) = -k_4\tilde{x}_1\,f_2(\tilde{\mathbf{x}})\,. \qquad (28)$$

The control is relay-based of the type (7). The controlled Master system has the form (2).

We choose to apply the concept of the so called *local control* [3] for each-input control. The sense of the local control is that by this type of control the control function is not applied immedeately to the controlled system, but the system is let to run free and only when the system trajectory enters into a sphere in the state space with a center - the chosen for the stabilization fixed point and radius - R, defined by us, we will apply the control. We will say that the system is stabilized in the point $\mathbf{x}_6^*$ when the trajectory enters into another small sphere with the same center $\mathbf{x}_6^*$ and sufficiently small radius $r<R$.

V. Experimental Results

The results shown are for the synchronization scheme (25). The other synchronization variants between the Master and the Slave systems give similar results, the only difference is the length of the transient until the stabilization of the Master system. Fig.4 depicts the state space of the controlled Master system and the control (7) for first-input control with a control function (26). The initial conditions of the two systems are $\mathbf{x}(0) = [5 \quad 2 \quad 1]^T$ and $\tilde{\mathbf{x}}(0) = [6 \quad 1 \quad 0]^T$. The big and the small spheres of the local control are with radii R = 6 and r = 0.8, the control gain is k = -20. The system stabilizes in the point $\mathbf{x}_{61}^* = [7.65 \quad 7.26 \quad 18.67]^T$.

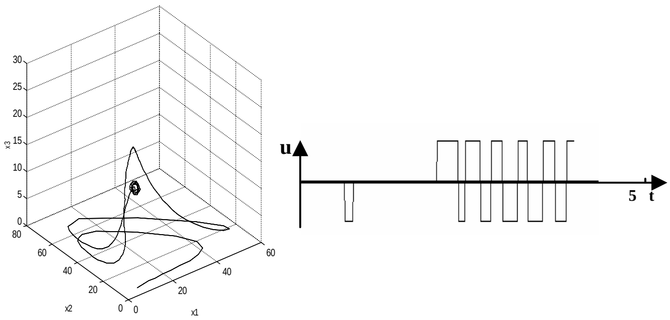

Fig.4. First input control - state space and control function u(t).

The same characteristics are shown on fig.5 for the second-input control with a control function (27). The systems are started from the same initial conditions. The spheres are with radii R = 6 and r = 0.3, the control gain is k = 10. The system stabilizes in the point $\mathbf{x}_{62}^* = [7.6 \quad 7.27 \quad 18.1]^T$.

Fig.6 shows the state space and the control for the third-input control with a control function (28) and the same initial conditions. The spheres are with radii R = 6 and r = 0.5, the control gain is k = -20. The system stabilizes in the point $\mathbf{x}_{63}^* = [6.87 \quad 7.45 \quad 17.94]^T$.

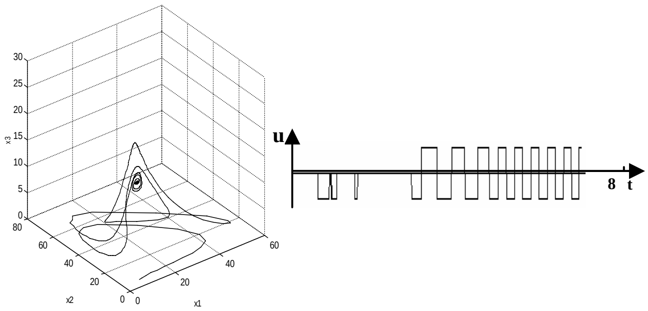

Fig.5. Second input control - state space and control function u(t).

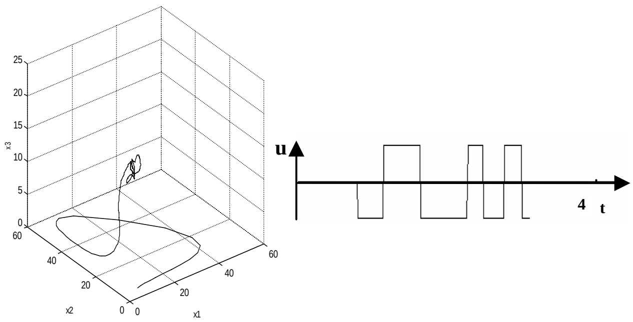

Fig.6. Third input control - state space and control function u(t).

VI. Conclusion

In this article an approach for the stabilization of chaotic systems, based on the phenomenon chaotic synchronization, is proposed. We also propose a synchronization method, which offers a great number of variants of connection between the Master and the Slave systems and by some of them the connection is only by one variable. This gives us the opportunity to choose the best variant in terms of speed of synchronization and starting from it, to realize a proper control to the Master system on the basis of the maximum principle, which will stabilize the system into a preliminarily chosen unstable fixed point.

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