The Probability Stability of Continuous Systems with Randomly Selected Parameters

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***Abstract* – An approximate method for the probability stability estimation of systems whose parameters are random values is presented in this paper. This method can be applied to the continuous and discrete systems as well. In the paper the method is illustrated using the example of the second order electrical circuit. Electrical circuit is modeled using bond graphs wherewith obtaining the characteristic equation of the system is enabled. The characteristic equation of the system is essential for the calculation of the probability stability. The computation results are shown in table.**

***Keywords* – Probability stability, continuous systems, bond graph.**

I. Introduction

In many branches of industry the systems in which some parameters cannot be computed and measured precisely occur. It happens that the values of the parameters do not coincide with the nominal values because these parameters often depend on values having stochastic character. For that influence the properties of the system deviate from the desirable values. Those deviations can be very large and can make the normal work of the system impossible. That is why is necessary to estimate the influence of the stochastic values on the system properties. This estimation is very important for the system stability and quality working.

The basic methods for the probability stability estimation of the randomly selected parameters are given in [1-4]. These methods relate to the continuous systems. However, the method can be applied on discrete systems with the random parameters too, [5-7].

The calculation of the probability stability enables the determining of such parameter values for which the system has the largest probability stability. The advantage of this method is in its practical use. In this paper the presented method is illustrated on the electrical circuit whose parameters are random, exponentially distributed values. The presented method can be applied to the other probability distributions also.

The characteristic equation of the system is essential for the calculation of the probability stability. There are different methods for deriving characteristic equation. In this paper the causal bond graphs are used for obtaining the state space model and characteristic equation of the system, [8-10].

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II. The Probability Stability Determining of the Continuous Systems

Let the transfer function of the continuous system is given by:

$$W(s) = \frac{s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \tag{1}$$

The appropriate characteristic equation of this system is:

$$s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0 \tag{2}$$

The coefficients of the characteristic equation are random variables with the probability density distributions $p_i(a_i)$. In the case when the system parameters are constant, system can be stable or non stable depending on parameter value. If the parameters of the system are random variables, the system can be stable with probability stability P. The main goal is to determinate that probability stability.

First the area stability of the system (1) must be obtained using some of the methods for the stability test, for example the Hurwitz criterion.

The system (1) is stable if all zeroes of the characteristic equation (2) are in the left half of the s –plane. The necessary and sufficient condition for the stability of system (1) is that all diagonal minors D_i of matrix D:

$$D = \begin{vmatrix} a_{n-1} & a_{n-3} & a_{n-5} & \cdots & 0 \\ a_n & a_{n-2} & a_{n-4} & \cdots & 0 \\ 0 & a_{n-1} & a_{n-3} & \cdots & 0 \\ 0 & a_n & a_{n-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 \end{vmatrix} \tag{3}$$

are greater than zero, i.e.:

$$D_1 = a_{n-1} > 0;$$

$$D_2 = \begin{vmatrix} a_{n-1} & a_{n-3} \\ a_n & a_{n-2} \end{vmatrix} > 0, \text{ etc} \tag{4}$$

The stability area of the continuous system (1) is obtained in the parametric space using the nonlinearities (4).

For the second order system $s^2 + a_1 s + a_0 = 0$ the stability area, S_2, is the first quadrant and it is defined by the next relations:

$$\begin{aligned} & a_1 > 0 \\ & a_1 \cdot a_0 > 0 \Rightarrow a_0 > 0 \end{aligned} \tag{5}$$

The stability area, S_3, for the third order continuous system $s^3 + a_2 s^2 + a_1 s + a_0 = 0$ is obtained from the next set of nonlinearities:

$$\begin{aligned} & a_2 > 0 \\ & a_1 \cdot a_2 - a_0 > 0 \Rightarrow a_1 > 0 \\ & a_0 \cdot (a_1 \cdot a_2 - a_0) > 0 \Rightarrow a_0 > 0 \end{aligned} \tag{6}$$

For the n – th order continuous system the area of stability is determined using the relations (4), also. However, the calculation is far too complex. Particularly, the limits of the stability area are usually complex mathematical relations and is difficult to determinate the probability stability because it is necessary to integrate by the area of stability.

If the probability density distributions $p_i(a_i)$ are given and if parameters of the systems are independent variables, then the total density distribution is given by:

$$p(a_1, \ldots, a_n) = \prod_{i=1}^{n} p_i(a_i) \tag{7}$$

The probability stability of the system (1) is:

$$P = \int_{S_n}\cdots\int p(a_1, \ldots, a_n)\, da_1 \cdots da_n \tag{8}$$

For the calculation of probability stability the knowledge of the system characteristic equation is necessary. Further in the paper the derivation of the characteristic equation using bond graphs will be presented.

III. Obtaining the System Characteristic Equation using Bond Graphs

Bond graphs, introduced in 1961 by Paynter, are very actual nowadays as universal approach for modeling of different types of physical systems. The main advantage of this modeling technique is that it is based on the fundamental physical law – the law of the energy conservation. Bond graph consists of elements exchanging energy through the connection connected them. These connections are bonds. The bond is presented with the half arrow indicating the direction of the energy flow between connected bond graph elements. The transported power is the product of two variables, the effort and the flow. The effort (for example: voltage, force, pressure, etc.) and the flow (for example: current, velocity, volume flow rate, etc.) are generalization of similar phenomenon in physics.

The important advantage of bond graphs is the natural selection of the variables. The memory of the system is in I and C bond graph elements, so it is natural to connect state variables to these elements. This connection, for the effort storage, I element, is done by mathematical relation describing this element:

$$f(t) = \frac{1}{\alpha}\int e(\tau)\,\mathrm{d}\tau \tag{9}$$

It is obvious to select the state variable *x* as:

$$x = f \Rightarrow \dot{x} = \frac{1}{\alpha} e \tag{10}$$

or:

$$x = \alpha f \Rightarrow \dot{x} = e \tag{11}$$

The both selections are equal, but only one of them can be natural from the physical point of view. In both cases, the flow f is determinated by x and the effort e is obtained by system state and input.

For the flow storage, *C* element, the appropriate equation is:

$$e(t) = \frac{1}{\beta}\int f(\tau)\,\mathrm{d}\tau \tag{12}$$

with the next selection of the state:

$$x = e \Rightarrow \dot{x} = \frac{1}{\beta} f \tag{13}$$

or:

$$x = \beta e \Rightarrow \dot{x} = f \tag{14}$$

Hence, in this case the state variable is the effort *e*, while the flow *f* can be obtained from the system state and the input.

If we introduce the state equations in Eqs. (10) and (13) for all *I* and *C* elements, the system description in the state space is obtained as well as all flows of *C* elements and all efforts of *I* elements as functions of states and inputs. The flow of *I* element and the effort of *C* element are state variables and the outputs of these elements as well. Therefore, the calculations can be done considering that *I* elements are flow sources, and *C* elements are effort sources. Further, efforts and flows of other bond graph elements in bond graph can be obtained knowing their mathematical relations.

The application of bond graphs for system modeling and obtaining the characteristic equation will be illustrated on example of the second order electrical circuit. On Fig. 1 the second order electrical circuit is shown.

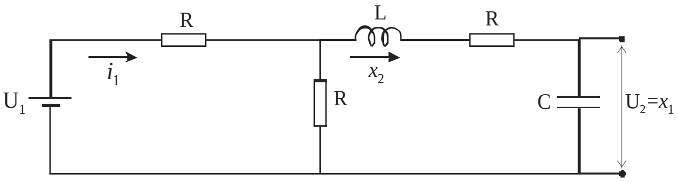

Fig.1. The second order electrical circuit

On Fig. 2 the causal bond graph model of given system is presented. This model is important for obtaining the state space equation.

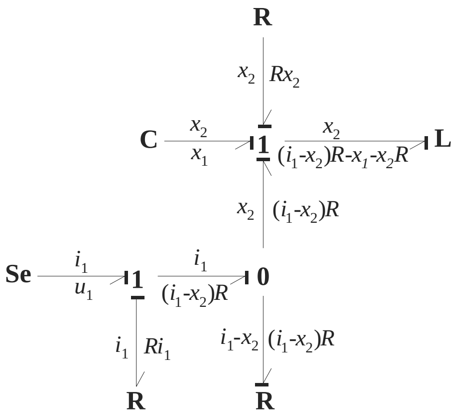

Fig.2. Causal bond graph model of the second order electrical circuit

In order to obtain the state space model, state variables are: effort on C element x_1 and flow through L element x_2. The state model is given by:

$$\dot{x}_1 = \frac{1}{C}x_2$$
$$\dot{x}_2 = \frac{1}{L}\left(\left(i_1 - x_2\right)R - x_1 - x_2R\right) \quad (15)$$

where current i_1 is:

$$i_1 = \frac{u_1 + Rx_2}{2R} \quad (16)$$

The state model becomes:

$$\dot{x}_1 = \frac{1}{C}x_2$$
$$\dot{x}_2 = -\frac{1}{L}x_1 - \frac{3}{2}Rx_2 + \frac{1}{2L}u_1 \quad (17)$$

The state matrix is given by:

$$\mathbf{A} = \begin{bmatrix} 0 & \frac{1}{C} \\ \frac{-1}{L} & \frac{-3R}{2L} \end{bmatrix} \quad (18)$$

The characteristic equation of the system is obtained from the next relation easily:

$$\det\left(s\mathbf{I} - \mathbf{A}\right) = 0 \quad (19)$$

where $\mathbf{I}$ is unit matrix of the second order and the characteristic equation is:

$$s^2 + \frac{3R}{2L}s + \frac{1}{LC} = 0 \quad (20)$$

IV. The Calculation Results

Comparing Eqs. (20) and (2) can be concluded that $a_1 = \frac{3R}{2L}$ and $a_0 = \frac{1}{LC}$. Assume that the resistance R and capacitance C change their values by the exponential law and the appropriate density distributions are $p_R = Ae^{-aR}$ and $p_C = Be^{-bC}$. The probability stability of the given electrical circuit due to the change of the parameter values should be determinate. Since a_1 and a_0 are the functions of the variables R and C, the transformation of this random variables must be done. Using the Jacobian:

$$J = \begin{vmatrix} \frac{\partial R}{\partial a_1} & \frac{\partial R}{\partial a_0} \\ \frac{\partial C}{\partial a_1} & \frac{\partial C}{\partial a_0} \end{vmatrix} \quad (21)$$

the next density distribution of the two – dimensional random variable is obtained:

$$p\left(a_1, a_0\right) = p\left(R, C\right) \cdot \left|J\right| \quad (22)$$

The probability stability of the system is obtained using Eq. (8) where S_2 is the first quadrant. The computation results are given in the next table. The resistance R does not influence on the computational results.

TABLE I
COMPUTATIONAL RESULTS

a	0.05	0.05	0.08	0.03	0.07	0.05
b	0.06	0.06	0.09	0.043	0.08	0.06
L $[H]$	1.5	1.1	0.6	0.2	0.2	0.1
P	0.329	0.424	0.776	0.883	0.981	0.997

V. Conclusion

Presented method enables the probability stability determining of the continuous systems with randomly selected parameters. This method can be used in practice which is demonstrated on the example of the second order electrical circuit. Electrical circuit is modeled using bond graphs wherewith the obtaining the characteristic equation of the system is enabled. The results are given in table. Using this method is possibly to choose such values of parameters for which the system has the largest probability stability.

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