Calculation of the Sampling Losses for Nonuniformly Sampled Data

Rossen G. Miletiev[1], Slavcho Lishkov[2]

Abstract - **A new definition of the optimum nonuniform sampling sequence is proposed. This definition is based on the spectrum analysis of nonuniformly sampled data according to Koh-Wicks-Sarkar spectrum estimation equation. Also simulation results are represented to calculate the sampling losses in comparison with the uniform sampling scheme.**

Keywords - **spectrum analysis, nonuniformly sampled data**

I. Introduction

Nonuniformly sampled data occurs in several applications such as geophysics [1], Laser Doppler Anemometry (LDA) [2], oscilloscopes [3] and radar or sonar signal processing [4]-[5]. Such type of data is used by the system designers to avoid aliasing in the signal spectrum or due to the technical problems, it is sometimes impossible to perform regular sampling.

Several methods for spectrum analysis of the nonuniformly sampled data is proposed, such as Lomb periodogram [6], Koh-Wicks-Sarkar equation [5], Dirichlet transformation [7], SECOEX method [8], non-uniform DFT [9] and some approximation methods [10]-[11]. But only few publications examine closely the problem with the optimum sampling sequences based on these spectrum estimation methods. These publications study the effect of the sampling scheme on the estimation performance and define the optimum sampling sequence for the alias frequency suppression to obtain low-aliasing spectrum estimation methods.

This paper studies the sampling losses of the nonuniformly sampled data depending on the sampling scheme and defines an equation for the optimal sampling sequence.

II. Mathematical Background

Lets a continuous complex signal $x(t)$ is sampled at time instants, $t=t_k$, $k=0,1,..,N$-1. The frequency response $E(j\omega)$ at frequency ω of the sequence is estimated by the equation [5]:

$$E(j\omega)=\frac{\sum_{k=0}^{N-1}x(t_k)\cos\omega(t_k-\tau)}{\sqrt{\sum_{k=0}^{N-1}\cos^2\omega(t_k-\tau)}}+j\frac{\sum_{k=0}^{N-1}x(t_k)\sin\omega(t_k-\tau)}{\sqrt{\sum_{k=0}^{N-1}\sin^2\omega(t_k-\tau)}} \qquad (1)$$

where τ is a free parameter, defined as:

$$\tan(2\omega\tau)=\frac{\sum_{k=0}^{N-1}\sin 2\omega t_k}{\sum_{k=0}^{N-1}\cos 2\omega t_k} \qquad (2)$$

The equation (2) can be written in the following way:

$$\sum_{k=0}^{N-1}\sin 2\omega(t_k-\tau)=0\cdot \qquad (3)$$

If we note the expression $\mu=\sum_{k=0}^{N-1}\cos 2\omega(t_k-\tau)$ and substitute the equation (3) in this expression, then the following equations can be written:

$$\sum_{k=0}^{N-1}\sin^2\omega(t_k-\tau)=\frac{1}{2}(N-\mu) \qquad (4)$$

$$\sum_{k=0}^{N-1}\cos^2\omega(t_k-\tau)=\frac{1}{2}(N+\mu) \qquad (5)$$

$$\sum_{k=0}^{N-1}\sin\omega(2t_k-\tau)=\mu\sin\omega\tau \qquad (6)$$

$$\sum_{k=0}^{N-1}\cos\omega(2t_k-\tau)=\mu\cos\omega\tau \qquad (7)$$

In this case we define the optimal sampling sequence as the time sampling sequence, which sets the spectrum peaks of the input signal to its maximum value.

If the signal contains the frequency component ω_0, then the optimal sampling sequence satisfies the following equation:

$$|E(\omega_0)|^2=\max \qquad (8)$$

So, if we perform the input complex signal $x(t)$ as a sum of two signals $x(t)=x_1(t)+jx_2(t)=\cos\omega_0 t+j\sin\omega_0 t$, then the frequency response of the input signal is a sum of the partial frequency responses:

$$E(\omega_0)=E_1(\omega_0)+jE_2(\omega_0) \qquad (9)$$

Each partial frequency response can be estimated by the equation (1). So if we substitute the equations (4) – (7) in the partial frequency responses $E_1(\omega_0)$ and $E_2(\omega_0)$, then the final expressions can be written as:

$$E_1(\omega_0)=\frac{1}{\sqrt{2}}\sqrt{N+\mu}\cos\omega_0\tau-j\frac{1}{\sqrt{2}}\sqrt{N-\mu}\sin\omega_0\tau \qquad (10)$$

$$E_2(\omega_0)=\frac{1}{\sqrt{2}}\sqrt{N+\mu}\sin\omega_0\tau+j\frac{1}{\sqrt{2}}\sqrt{N-\mu}\cos\omega_0\tau\cdot \qquad (11)$$

Therefore, the frequency response of the input signal can be estimated by equations (9) - (11):

$$E(\omega_0)=\frac{1}{\sqrt{2}}\cos\omega_0\tau\left(\sqrt{N+\mu}-\sqrt{N-\mu}\right)+ +j\frac{1}{\sqrt{2}}\sin\omega_0\tau\left(\sqrt{N+\mu}-\sqrt{N-\mu}\right) \qquad (12)$$

The power spectrum can be estimated by the equation (12):

$$|X(\omega_0)|^2=N+\sqrt{N^2-\mu^2} \qquad (13)$$

[1]Rossen Miletiev is with the Technical University of Sofia, 8 Kl. Ohridski Blvd, 1000 Sofia, Bulgaria, E-mail: miletiev@yahoo.com

[2]Slavcho Lishkov is with the Technical University of Sofia, 8 Kl. Ohridski Blvd, 1000 Sofia, Bulgaria

So, the optimal sampling scheme satisfies the equation:

$$\mu = 0. \tag{14}$$

So, the optimal sampling scheme is up to the following requirements according to equations (3) and (14):

$$\begin{cases} \sum_{k=0}^{N-1} \cos 2\omega_0 t_k = 0 \\ \sum_{k=0}^{N-1} \sin 2\omega_0 t_k = 0 \end{cases} \tag{15}$$

The uniform sampling sequence defines the time samples according to the equation $t_k=kT$, which modifies the equation (15) to the following term:

$$\begin{cases} \sum_{k=0}^{N-1} \cos 2\omega_0 t_k = \dfrac{\sin N\omega_0 T}{\sin \omega_0 T} \cos(N-1)\omega_0 T \\ \sum_{k=0}^{N-1} \sin 2\omega_0 t_k = \dfrac{\sin N\omega_0 T}{\sin \omega_0 T} \sin(N-1)\omega_0 T \end{cases}, \tag{16}$$

where $\omega_0 = 2\pi k f_0 = 2\pi k \, {}^{f_s}\!/_{N}$.

Therefore, the uniform sampling scheme is optimal according to equation (15), because it satisfies the both trigonometric sums for the frequencies, defined by the Fourier transform.

The nonuniform sampling case is difficult to be analyzed because the equation (15) can not be solved in the general case. By reason of this circumstance, we will analyze the sampling losses, which are generated by the deviation of the used sampling sequence from the optimal one.

To analyze the sampling losses, we assume the sampling scheme, shown at Fig.1.

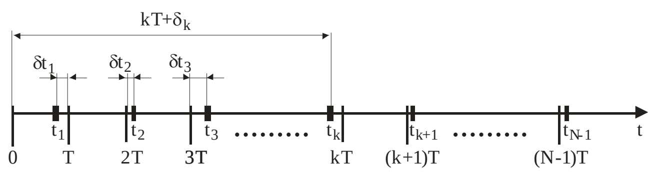


Fig.1. Sampling diagram

The sampling losses are evaluated by the equation:

$$\varepsilon = \frac{|X(\omega_0)|^2}{|X(\omega_0)|^2_{\max}} = \frac{1+\sqrt{1-\left(\dfrac{\mu}{N}\right)^2}}{2} \tag{17}$$

Therefore, the sampling losses vary from 0dB to 3dB according to parameter value ${}^{\mu}\!/_{N}$ (Fig.2).

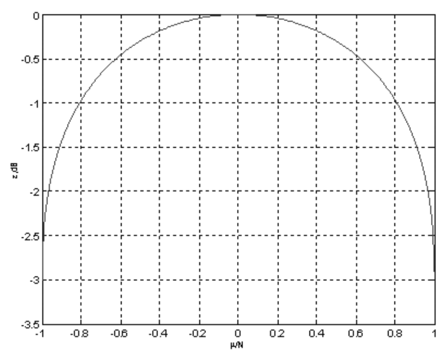


Fig.2. Sampling losses vs. parameter ${}^{\mu}\!/_{N}$

The shown figure displays that the sampling losses can be neglected if the parameter value ${}^{\mu}\!/_{N}$ is smaller than 0,3. In this case the losses are smaller than 0,1dB, which defines such types of sampling schemes as quasi-optimal ones.

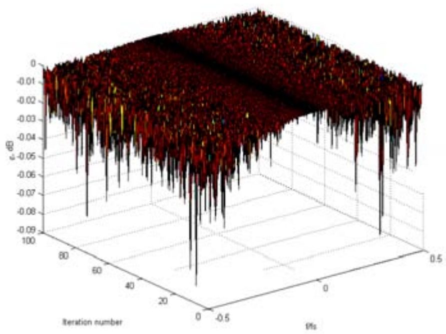
When the parameter value ${}^{\mu}\!/_{N}$ exceeds this limit, the losses increased very fast and they are equal to 1dB when the analyzed parameter is set to 0,8.

III. Simulation Results

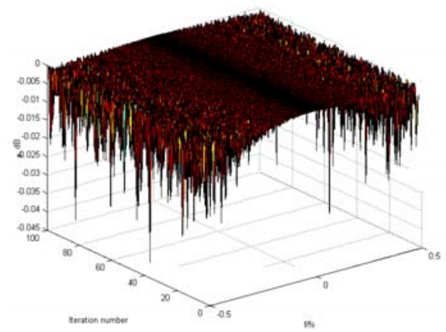
The influence of the sampling sequence over the sampling losses is calculated by simulation using MATLAB routine. We implement three times per 100 independent generation of sampling instants δ_k, which are uniformly distributed in the following intervals:

- $\delta_k \le {}^{1}\!/_{2} T$
- $\delta_k \le {}^{1}\!/_{4} T$
- $\delta_k \le {}^{1}\!/_{8} T$.

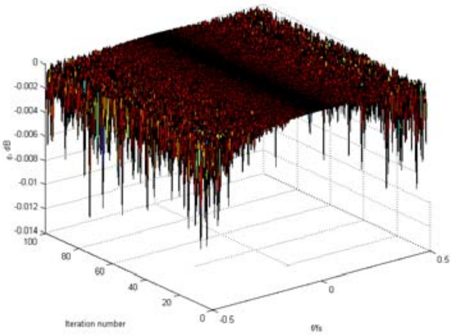
Then we calculate the sampling losses in these three simulations according to equation (17) using equations (4)-(7) below and above Nyquist limit for each case.



(a)



(b)

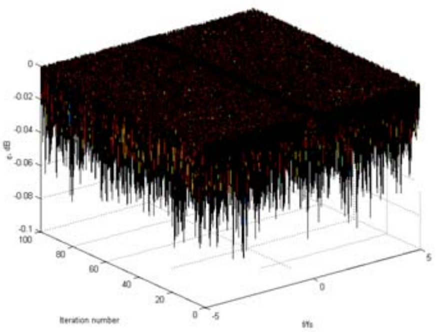


(c)

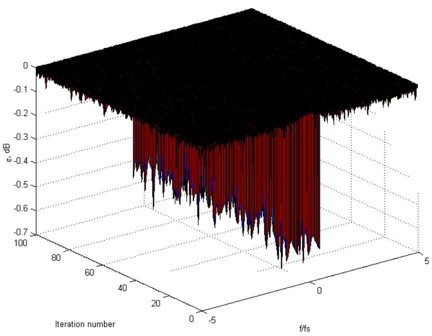
Fig.3. Sampling losses below Nyquist limit

The simulation results below Nyquist frequency in these three cases are shown at Fig.3a-c respectively. The simulation results show that the maximum sampling losses are reduced from 0,1dB to 0,01dB while the sampling instants δ_k are decreased from $^1/_2$T (Fig.3a) to $^1/_8$T (Fig.3c).

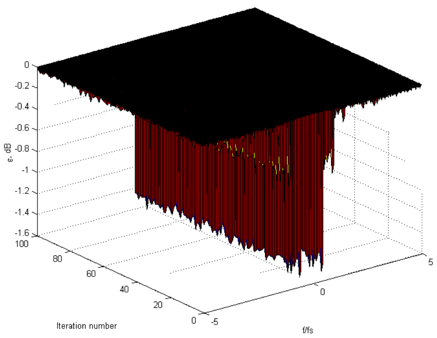
The simulation results are totally different when we calculate the sampling losses above Nyquist limit. In this case the simulation results are represented at Fig.4a-c for the chosen sampling instants intervals.



(a)



(b)



(c)

Fig.4. Sampling losses above Nyquist limit

The shown figures display that the sampling losses remain unchanged when the analyzed frequencies differ from the values $f = {}^{kf_s}\!/_2 = {}^{k}\!/_{2T}, k = \pm1,\pm2,....,\pm\infty$.

When the chosen frequencies are much closed to these values, the sampling losses increased very quickly from 0,1dB to 1,6dB while the sampling instants δ_k are decreased from $^1/_2T$ (a) to $^1/_8T$ (c).

The represented results show that the sampling losses remain below 0,1dB while the sampling intervals satisfy the equation $\delta_k \leq {}^1/_2T$. These losses are nearly independent from the chosen sampling intervals.

IV. Conclusion

This paper studies the sampling losses of the nonuniformly sampled data depending on the sampling scheme and defines an equation for the optimal sampling sequence.

The sampling losses are calculated according to the sampling interval choice and an optimal nonuniform sampling scheme equation is defined. The simulation results show that the sampling losses are very small and they are nearly independent from the chosen sampling scheme while the sampling intervals do not exceed $^1/_2T$ limit.

Therefore, this sampling scheme is recommended due to its low sampling losses and low aliasing frequency response at very wide frequency range.

References

[1] P. Yuou, E. Baert and M. F. Loutre, "Spectral analysis of climate data," *Surv. Geophys.*, vol. 17, pp. 619–663, 1996.

[2] C. Tropea, "Laser doppler anemometry: Recent developments and future challenges," *Meas. Sci. Technol.*, vol. 6, pp. 605–619, 1995.

[3] Y. Rolain, J. Schoukens, and G. Vandersteen, "Signal reconstruction for non-equidistant finite length sample sets: A "KIS" approach," *IEEE Trans. Instrum. Meas.*, vol. 47, pp. 1046–1052, 1998.

[4] R.Pribic, "Radar irregular sampling," *ICASSP 2004*, vol.3, pp.933-936, 2004.

[5] J. Koh, T. Sarkar and M. Wicks, 'Spectral analysis of nonuniformly sampled data using a least square method for application in multiple PRF system," *Proceedings of IEEE International Conference on Phased Array Systems and Technology 2000*, pp. 141 -144

[6] N. Lomb, "Least square frequency analysis unequally sampled data," *Aerophysics and space science*, vol.39, pp.447-462, May, 1975

[7] M. Tuszynski and A. Wojtkiewicz, "Application of the Dirichlet transform in analysis of nonuniformly sampled signals," 1992 IEEE International Conference on Acoustics, Speech, and Signal Processing ICASSP-92, May 23-26, 1992, San Francisco, USA, Vol. 5, pp.25-28

[8] W. Chen, G. Zhou and G. Giannakis, "Velocity and acceleration estimation of doppler weather radar/lidar signals in coloured noise," *IEEE International Radar Conference*, 1995, pp.2052-2055

[9] Qing Huo Liu and Nhu Nguyen, "An accurate algorithm for nonuniform fast Fourier transform (*NUFFT's*)," *IEEE Microwave and guided wave letters*, Vol.8, No.1, 1998, pp.18-20

[10] D. Bland, T. Laakso and A. Tarczynski, "Analysis of algorithms for nonuniform-time discrete Fourier transform," *IEEE International Symposium on Circuits and Systems* ISCAS '96, Vol. 2 , 12-15 May 1996, pp: 453 -456

[11] D. Bland, T. Laakso and A. Tarczynski, "Spectrum estimation of non-uniformly sampled signals," *Proceedings of the IEEE International Symposium on Industrial Electronics*, ISIE '96, Vol.1, 17-20 Jun 1996, pp. 196 -200