Bennett’s Integral for Uniform Polar Quantization

Zoran H. Peric[1], Jelena R. Nikolić[2], Danijela R. Aleksić[3]

Abstract - **In this paper the analysis of Bennett's integral is given for uniform polar quantization for two-dimensional memoryless Gaussian sources with respect to granular distortion D_g, i.e. the mean-square error (MSE). The goal of this paper is finding simple equation for distortion solving Bennett's integral for uniform polar quantization and circular symmetric sources (iid Gaussian source).**

Keywords – **Bennett’s integral, uniform polar quantization, normalized moment of inertia**

I. Introduction

The most important results in polar quantization was given by Swaszek and Ku who derived the asymptotically Unrestricted Polar Quantization (UPQ) [1]. Using asymptotic uniform polar analysis for calculating optimal numbers of magnitude levels and the numbers of reconstruction points for two-dimensional quantizers is done in [1, 2]. In [3] was given the analysis of vector quantization in order to determine the optimal maximal amplitude. In papers [4] and [5] were given analyses for asymptotic uniform polar quantization and asymptotic nonuniform polar quantization, respectively. Optimisatons was done with respect to granular distortion D_g, i.e. the mean-square error (MSE). In this paper Bennett's integral has been considered and succesful attempt has been made in order to extend it to the polar quantizer. Considerable attention has been focused on finding simple approximate formula for distortion that shows influence of key characteristics.

II. Uniform Polar Quantization

For this analysis we use uniform polar quantizer with L madnitude levels and M_i phase reconstruction levels at the magnitude reconstruction levels m_i, $1\le i\le L$. First we portion the magnitude range $[0, r_{L+1}]$ into magnitude rings with $L+1$ decision levels $\boldsymbol{r}=(r_1,\dots,r_{L+1})$ ordered in turn such that $0=r_1<r_2<\dots<r_L<r_{L+1}=r_{max}$. The magnitude reconstruction levels $\boldsymbol{m}=(m_1,\dots,m_L)$ obviously satisfy succession $0<m_1<m_2<\dots<m_L$. Let we assume that the total number of reconstruction points N is large enough. In that case magnitude decision levels and reconstruction levels are given in turn:

$$r_i=(i-1)\Delta, \qquad 1\le i\le L+1 \tag{1}$$

$$m_i=(i-1/2)\Delta, \qquad 1\le i\le L\ . \tag{2}$$

Let we consider distortion D as a function of the vector $P=(P_i)_{1\le i\le L}$ whose elements are values of phase quantization levels at the each magnitude levels. Said in other words, each concentric ring in quantization pattern is allowed to have a different number of partitions in the phase quantizer (P_i) for r in the i-th magnitude ring. Assuming that the representation points are centered in their respective cells, magnitude decision levels and reconstruction levels can be given as in Eqs. (1) and (2). Let we make a partition of each magnitude ring into M_i phase subpartitions. By denoting adjacent phase decision levels with $\phi_{i,j}$ and $\phi_{i,j+1}$, and the j-th phase reconstruction levels as $\psi_{i,j}$ for the i-th magnitude ring, $1\le j\le M_i$, following dependence is valid:

$$\phi_{i,j}=(j-1)\frac{2\pi}{P_i}. \tag{3}$$

Total distortion D may be written as the sum of granular and overload distortions, $D=D_g+D_o$. The granular distortion D_g can be given by:

$$D_g=\sum_{i=1}^{L}\sum_{j=1}^{P_i}\int_{\phi_{i,j}}^{\phi_{i,j+1}}\int_{r_i}^{r_{i+1}}[r^2+m_i^2-2rm_i\cos(\phi-\psi_{i,j})]\frac{f(r)}{2\pi}dr\,d\phi\ . \tag{4}$$

For this analysis we assume that the input is from a continuously valued circularly source with unit variance rectangular coordinate marginals and bivariate density function:

$$f(x,y)=p(\sqrt{x^2+y^2}). \tag{5}$$

Transforming to polar coordinates, the phase is uniformly distributed on $[0,2\pi)$ and the magnitude is distributed on $[0,\infty)$ with density function $f(r)=2\pi rp(r)$. The magnitude and phase are independent random variables. The transformed probability density function for the Gaussian source is:

$$f(r,\phi)=\frac{1}{2\pi\sigma^2}re^{-\frac{r^2}{2\sigma^2}}=\frac{f(r)}{2\pi} \tag{6}$$

[1]Zoran H. Perić is with the Faculty of Electronic Engineering, University of Nis, Beogradska 14, 18000 Nis, Serbia
E-mail: peric@elfak.ni.ac.yu
[2]Jelena R. Nikolić is with the Faculty of Electronic Engineering, University of Nis, Beogradska 14, 18000 Nis, Serbia
E-mail: njelena@elfak.ni.ac.yu
[3]Danijela R. Aleksić is with Telekom Serbia, Mobile Devision, 18000 Nis, SerbiaE-mail: danijelaal@telekom.yu

without lossing generality we assume that variance is $\sigma^2=1$. Suppose that a polar quantizer has many cells which are small and the source density is smooth. In that case granular distortion D_g of one cell is given by:

$$D_{i,j} = \frac{1}{2}\int_{R_{i,j}} \left[r^2 + m_i^2 - 2rm_i\cos\left(\phi - \psi_{i,j}\right)\right]\frac{f(r)}{2\pi}drd\phi \quad (7)$$

$$D_{i,j} = \frac{1}{2}\frac{f(m_i)}{2\pi}\int_{R_{i,j}} \left[r^2 + m_i^2 - 2rm_i\cos\left(\phi - \psi_{i,j}\right)\right]drd\phi\,. \quad (8)$$

The total granular distortion for polar quantization was found in [4]:

$$D_g^{pol} = \sum_{i=1}^{L}\sum_{j=1}^{P_i} D_{i,j} = \sum_{i=1}^{L} P_i D_{i,j} \quad (9)$$

$$D_g^{pol} = \sum_{i=1}^{L}\frac{\Delta}{12}\left(\frac{\Delta^2}{2} + \frac{2m_i^2\pi^2}{P_i^2}\right) f(m_i)\,. \quad (10)$$

III. BENNETT'S INTEGRAL

A two-dimensional N-point scalar quantizer is characterized by a partition $S=\{S_1,S_2,..., S_N\}$ of two-dimensional Euclidean space R^2 into N quantization cells and code book, noticed as $C=\{y_1,y_2,...,y_N\}$ is consisting of N quantization points in two-dimensional Euclidean space R^2. A two-dimensional source vector $\boldsymbol{x}=\{x_1,x_2\}$ is quantized into one of the y_i's according to the quantization rule $Q(x)=y_i$ if $x \in S_i$. Encoding rate for two-dimensional quantizer, is $log_2 N/2$. When applied to a random vector $\boldsymbol{x}=\{x_1,x_2\}$ with probability density $p(\boldsymbol{x})$, quantizer's distorsion can be given by:

$$D(S,C) = \frac{1}{2}\int \left\|\mathrm{x} - Q(\mathrm{x})\right\|^2 p(\mathrm{x})d\mathrm{x} \quad (11)$$

$$D(S,C) = \frac{1}{2}\sum_{i=1}^{N}\int_{S_i}\left\|\mathrm{x} - \mathrm{y}_i\right\|^2 p(\mathrm{x})d\mathrm{x} \quad (12)$$

where $\|\boldsymbol{x}-\boldsymbol{y_i}\|$ denotes Euclidean distance such that:

$$\left\|\mathrm{x} - \mathrm{y}_i\right\| = \left(\sum_{j=1}^{2}\left(x_j - y_{ij}\right)^2\right)^{1/2} \quad (13)$$

and $p(\boldsymbol{x})$ is the two-dimensional density of $\boldsymbol{x}$.

Bennett showed that the mean-squared error of a scalar quantizer (k = 1), with meny small cells (N large) and with each y_i in the center of its cell may be accurately approximated as:

$$D(S,C) \cong \frac{1}{12N^2}\int \frac{1}{\lambda(x)^2}p(x)dx \quad (14)$$

where, $\lambda(x)$ is a function, called point density, and $\lambda(x)\Delta$ is the fraction of quantization points in a small interval of width Δ surrounding x. The integral without limits denotes an integral over the entire space. The right-hand side of previous equation is known as Bennett's integral. Althought originally derived for companders (quantizers consisting of a compressor, uniform quantizer, and expander) with λ equal to the derivative of the compressor function can be recognized by other quantizers, and is applied more generally. Bennett's integral shows how the distortion depends on the key characteristics of the quantizer, namely, the number of points N and a point density λ. Its utility is exemplified by the fact that one may use it to show that the best quantizers have:

$$\lambda(x) \cong \frac{p(x)^{\frac{1}{3}}}{\int p(x)^{\frac{1}{3}}dx}\,. \quad (15)$$

In following analysis we extend Bennett's integral to polar quantizer. We suppose that distortion for polar quantization may be approximated as:

$$D(S,C) = \sum_{i=1}^{L}\sum_{j=1}^{P_i} p\left(m_i,\psi_{i,j}\right)NM(i)vol^2\left(S_{i,j}\right) \quad (16)$$

where $vol(S_{i,j})$ denotes the volume of the cell S_i and $NM(i)$ denotes the normalized moment of inertia of the cell S_i about the point y_i with respect to the distorsion and can be expressed in turn:

$$vol\left(S_{i,j}\right) = rdrd\varphi = \frac{\left(r_{i+1}^2 - r_i^2\right)\pi}{P_i} = \frac{2m_i\Delta_r\pi}{P_i} \quad (17)$$

$$NM(i) = \frac{1}{12}\left(\frac{P_i\Delta_r}{4m_i\pi} + \frac{m_i\pi}{\Delta_r P_i}\right) \approx \frac{1}{12}\left(\frac{p(r)\Delta_r}{4r\pi} + \frac{r\pi}{\Delta_r\, p(r)}\right) = m(r,\varphi)\,. \quad (18)$$

The width of rings, noticed with Δ_r, in case of resticted uniform polar quantization is $\Delta_r = r_{max}/L$. Finally point density is found as:

$$\lambda(m_i,\psi_{i,j}) = \frac{1}{Nvol(S_{i,j})} \approx \frac{1}{N}\frac{p(r)L}{r_{\max}r2\pi} = \lambda(r,\varphi)\,. \quad (19)$$

After this approximation, D becomes:

$$D(S,C) = \int_0^{r_{\max}}\int_0^{2\pi} p_r(r,\varphi)\frac{m(r,\varphi)}{\lambda(r,\varphi)}rdrd\varphi$$

$$= \int_0^{r_{\max}} r e^{-\frac{r^2}{2}} \left(\frac{1}{24}\frac{r_{\max}^2}{L^2} + \frac{\pi r}{\Delta(p(r))^2}\right) dr . \qquad (20)$$

By using the method of Lagrange multipliers with restriction for the total number of the reconstruction points N we obtained optimal point density *p(r)* as:

$$p(r) = \frac{N r_{\max}}{3\left(1 - e^{-r_{\max}^2/6}\right)} r e^{-\frac{r^2}{6}} . \qquad (21)$$

Let distortion can be determinated as: $D(S,C) = I_1 + I_2$ where are:

$$I_1 = \int_0^{r_{\max}} \frac{r_{\max}^2}{24L^2} r e^{-\frac{r^2}{2}} dr =$$

$$= \frac{1}{24}\left(\frac{r_{\max}}{L}\right)^2 \left(1 - e^{-\frac{r_{\max}^2}{2}}\right) \approx \frac{1}{24}\left(\frac{r_{\max}}{L}\right)^2 \qquad (22)$$

$$I_2 = \int_0^{r_{\max}} \frac{\pi^2}{6(p(r))^2} r^3 e^{-\frac{r^2}{2}} dr =$$

$$= \frac{1}{2}\left[\frac{3\pi\left(1 - e^{-\frac{r_{\max}^2}{6}}\right)L}{N r_{\max}}\right]^2 \left(1 - e^{-\frac{r_{\max}^2}{6}}\right) . \qquad (23)$$

Finally we find distortion as a solution of Bennett's integral for uniform polar quantizer as:

$$D^{\dim}(S,C) = \frac{1}{24}\left(\frac{r_{\max}}{L}\right)^2 + \frac{1}{2}\left[\frac{3\pi\left(1 - e^{-\frac{r_{\max}^2}{6}}\right)L}{N r_{\max}}\right]\left(1 - e^{-\frac{r_{\max}^2}{6}}\right) \qquad (24)$$

The optimal number of levels is:

$$L = \frac{r_{\max}}{\sqrt[4]{108\left(1 - e^{-r_{\max}^2/6}\right)^3}} \sqrt{\frac{N}{\pi}} \qquad (25)$$

and we find distortion as:

The width of rings, noticed with Δ_r, in case of resticted uniform polar quantization is $\Delta_r = r_{\max}/L$. Finally point density is found as:

$$\lambda(m_i, \psi_{i,j}) = \frac{1}{N vol(S_{i,j})} \approx \frac{1}{N}\frac{p(r)L}{r_{\max} r 2\pi} = \lambda(r,\varphi) . \qquad (19)$$

After this approximation, D becomes:

$$D(S,C) = \int_0^{r_{\max}} \int_0^{2\pi} p_r(r,\varphi) \frac{m(r,\varphi)}{\lambda(r,\varphi)} r dr d\varphi$$

$$= \int_0^{r_{\max}} r e^{-\frac{r^2}{2}} \left(\frac{1}{24}\frac{r_{\max}^2}{L^2} + \frac{\pi r}{\Delta(p(r))^2}\right) dr . \qquad (20)$$

By using the method of Lagrange multipliers with restriction for the total number of the reconstruction points N we obtained optimal point density *p(r)* as:

$$p(r) = \frac{N r_{\max}}{3\left(1 - e^{-r_{\max}^2/6}\right)} r e^{-\frac{r^2}{6}} . \qquad (21)$$

Let distortion can be determinated as: $D(S,C) = I_1 + I_2$ where are:

$$I_1 = \int_0^{r_{\max}} \frac{r_{\max}^2}{24L^2} r e^{-\frac{r^2}{2}} dr =$$

$$= \frac{1}{24}\left(\frac{r_{\max}}{L}\right)^2 \left(1 - e^{-\frac{r_{\max}^2}{2}}\right) \approx \frac{1}{24}\left(\frac{r_{\max}}{L}\right)^2 \qquad (22)$$

$$I_2 = \int_0^{r_{\max}} \frac{\pi^2}{6(p(r))^2} r^3 e^{-\frac{r^2}{2}} dr =$$

$$= \frac{1}{2}\left[\frac{3\pi\left(1 - e^{-\frac{r_{\max}^2}{6}}\right)L}{N r_{\max}}\right]^2 \left(1 - e^{-\frac{r_{\max}^2}{6}}\right) . \qquad (23)$$

Finally we find distortion as a solution of Bennett's integral for uniform polar quantizer as:

$$D^{\dim}(S,C) = \frac{1}{24}\left(\frac{r_{\max}}{L}\right)^2 + \frac{1}{2}\left[\frac{3\pi\left(1 - e^{-\frac{r_{\max}^2}{6}}\right)L}{N r_{\max}}\right]\left(1 - e^{-\frac{r_{\max}^2}{6}}\right) \qquad (24)$$

The optimal number of levels is:

$$L = \frac{r_{\max}}{\sqrt[4]{108\left(1-e^{-r_{\max}^2/6}\right)^3}}\sqrt{\frac{N}{\pi}} \qquad (25)$$

and we find distortion as:

$$D^{\dim} = \frac{1}{2}\frac{\pi\sqrt{3}}{N}\left(1-e^{-\frac{r_{\max}^2}{6}}\right)^{\frac{3}{2}}. \qquad (26)$$

Optimal point density is $p(r)=2Nr/(r_{max}L)$ in case of $\lambda(r,\varphi)=const$, and optimal number of magnitude levels is $L=\sqrt{N/\pi}$. Hence, distortion can be found as:

$$D_{gconst}^{\dim} = \frac{r_{\max}^2 \pi}{12N}. \qquad (27)$$

Table I gives parallel comparison of distortions $D_{gconst}^{\dim}$ and $D_g^{\dim}$ when number of quantization cells are N=256, N=4096 and N=65536.

TABLE I

COMPARED RESULTS FOR $D_{gconst}^{\dim}$ AND $D_g^{\dim}$ FOR A DIFFERENT BIT RATES (R) AND NUMBER OF RECONSTRUCTION POINTS (N)

R	N	$D_{gconst}^{\dim}$	$D_g^{\dim}$
4	256	0.0106	0.00637
6	4096	0.00106	0.00026
8	65536	0.000093	0.00000856

IV. CONCLUSION

The analysis of Bennett's integral is given for uniform polar quantization for two-dimensional memoryless Gaussian sources. This paper gives simple and complete analysis for constructing an optimal uniform polar quantizer for sources with optimal point density. We have calculated granular distortion and have found gain obtained by using optimal point density. The goal of tis paper is solving quantization problems for uniform polar quantizers by finding minimal distortion and optimal point density.

REFERENCES

[1] P. F. Swaszek, T. W. Ku, “Asymptotic Performance of Unrestricted Polar Quantizer”, *IEEE Transactions on Information Theory*, vol.32, pp. 330-333, 1986.

[2] R. M. Gray and D. L. Neuhoff, “Quantization”, *IEEE Transactions on Information Theory*, vol.44, no.6, pp. 2325-2384, October 1998.

[3] S. Na D. L. Neuhoff, "Bennett’s Integral for Vector Quantizers," *IEEE Transaction on Information Theory*, vol.41, pp. 886-900, July 1995.

[4] Z. H. Perić, M. Č. Stefanović, “Asymptotic Analysis of Optimal Uniform Polar Quantization”, *International Journal of Electronics and Communications,* vol.56, pp. 345-347, 2002.

[5] D. Aleksić “Asymptotic Analysis of Nonuniform Polar Quantiyation”, Telfor, 2003.