Introduction of Heaviside Criterion Fulfillment Factor for Transmission Line Analysis

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Abstract –**In this paper, Heaviside criterion for TEM transmission line, that provides wave propagation along a line without dispersion, is presented. Through an introduction of Heaviside criterion fulfillment factor, an influence of per-unit length inductance increase on the transmission line behaviour is discussed. Limit values of characteristic propagation functions are derived and an analysis of their behaviour in the frequency domain for different values of proposed factor is done.**

Keywords – **transmission line, Heaviside criterion, fulfillment factor**

I. INTRODUCTION

The effect of loss on a transmission line causes attenuation and dispersion of a propagating wave. With dispersion, different frequencies in a signal spectrum propagate with different velocities and lead to distortion of any non-sinusoidal wave form. Both the attenuation and distortion resulting from loss in transmission lines were responsible for the delay in the development of long distance telephone communication of speech in the early days of the telephone. In 1893, Heaviside developed the transmission line theory based on Maxwell's equations [1], which made preconditions for a telephone transmission of human voice over a long distance. Until that time, the transmission line was described by a diffusion equation, which in circuit term, involves a distributed series resistor and paralell capacitor network.

Taking proper account of the inductance, Heaviside noticed that the effects of attenuation and distortion both decreased as inductance is increased. Analysing the frequency dependence of propagation constant, he derived the relation between the primary per-unit length parameters of transmission line (so-called Heaviside criterion) from the condition of minimum losses. When this criterion is fulfilled, both attenuation constant and velocity of propagation are independent of frequency. Transmission line with such primary parameters has an infinitive bandwidth assuming that only TEM wave propagates. In that case, an output signal is of the same waveform as an input signal; it is only attenuated and delayed in regard to the input signal.

As telecommunication lines generally have a small per-unit length inductance ($LG << RC$), Serbian scientist Mihajlo Pupin suggested that, in order to increase inductance, telephone lines should be loaded in series with discrete coils placed at regular intervals along a line [2,3]. In 1900, he experimentally verified the proposed approach in his laboratory at Columbia University in USA. This method, today well-known as Pupin loading, is still used in local and trunk telephone lines. At frequencies for which wavelength λ is a significantly greater than the spacing a of the loading coils ($\lambda >> \pi a$ [4]), pupinized line is behaved as an equivalent line obtained as if L were increased continuously along the line.

This paper represents a continuation of novel research work, presented in references [5,6], established with the main goal to enlighten the scientific achievements of Mihajlo Pupin. In this paper, an influence of per-unit length inductance increase on transmission line characteristics is presented. For that purpose, a new parameter - Heaviside criterion fulfillment factor is introduced. Also, the limit values of characteristic functions determining propagation along the line, for two end frequencies, $\omega \to 0$ and $\omega \to \infty$ are derived. Their behaviour in the frequency domain for different values of suggested factor is analysed and the appropriate conclusions are given.

II. PARAMETERS OF LOSSY TRANSMISSION LINE

Propagation constant of a lossy TEM transmission line can be expressed as [4]:

$$\gamma(\omega) = \alpha(\omega) + j\beta(\omega) = \sqrt{(R + j\omega L)(G + j\omega C)} \tag{1}$$

where: α is an attenuation constant in Np/m or dB/m, β is a phase constant in rad/m, R, L, G and C are primary per-unit length resistance, inductance, conductance and capacitance of transmission line, respectively and ω is a radial frequency in rad/s. Solving a system of two equations, obtained by equating the real and imaginary parts of left and right side of Eq.(1), for unknown α and β, leads to:

$$\alpha^2(\omega) = \frac{RG - \omega^2 LC}{2} + \sqrt{\left(\frac{\omega^2 LC - RG}{2}\right)^2 + \left(\frac{\omega(LG + RC)}{2}\right)^2} \tag{2}$$

$$\beta^2(\omega) = \frac{\omega^2 LC - RG}{2} + \sqrt{\left(\frac{\omega^2 LC + RG}{2}\right)^2 + \left(\frac{\omega(LG - RC)}{2}\right)^2} \tag{3}$$

How attenuation constant depends on frequency, different components in signal spectrum will be differently attenuated. This effect, known as amplitude distortion, is combated with the use of band-pass filters, which split the signal into various frequency bands at the output end. For each band different application factor A is used so that product $\alpha(\omega)A(\omega)$ is constant at all frequencies.

Characteristic impedance of lossy TEM transmission line is a complex quantity as well and it can be calculated as:

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$$Z_c(\omega) = |Z_c(\omega)| e^{j\varphi_c(\omega)} = \sqrt{(R + j\omega L)/(G + j\omega C)} \quad (4)$$

In telecommunications, a modulated signal is regularly propagated along the line, occupying a considerable frequency band. Two quantities can be used to define its velocity of propagation: the phase velocity, v_p, as a velocity of propagation for the carrier and the group velocity, v_g, as a velocity of propagation for the envelope of modulated signal or velocity with which energy is propagated along the line:

$$v_p(\omega) = \frac{\omega}{\beta(\omega)}, \; v_g(\omega) = \frac{d\omega}{d\beta(\omega)} \quad (5)$$

In reality, function $\beta(\omega)$ is not a straight line so that v_p and v_g are generally different from each other. Also, they vary with frequency. Since the group velocity represents velocity of propagation of the various components in the modulated signal frequency spectrum, the time taken for the components to be propagated along a line of given length will not be the same. Then, it would be impossible to reconstruct the spectrum of transmitted signal at output which leads to the so-called signal phase distortion. Such an effect, known as dispersion, is a great problem in telecommunications and it can be overcome by inserting expensive variable delay lines in each frequency band. For a unit length line, the phase and group delay are related to the phase and group velocity as:

$$\tau_p(\omega) = \frac{1}{v_p(\omega)} = \frac{\beta(\omega)}{\omega}, \; \tau_g(\omega) = \frac{1}{v_g(\omega)} = \frac{d\beta(\omega)}{d\omega} \quad (6)$$

Of possible interest for an analysis are values of previously defined parameters for $\omega \to 0$ and $\omega \to \infty$. To determine them, we start from the product of phase and group delay:

$$\tau_p(\omega)\tau_g(\omega) = \frac{\beta(\omega)}{\omega}\frac{d\beta(\omega)}{d\omega} = \frac{1}{2\omega}\frac{d\beta(\omega)^2}{d\omega} = \tau_{pg}(\omega) \quad (7)$$

Finding the derivation of Eq.(3) per ω, it can be obtained:

$$\tau_{pg}(\omega) = \frac{1}{4}\left(2LC + \frac{2\omega^2 L^2 C^2 + L^2 C^2 + R^2 C^2}{\sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)}}\right) \quad (8)$$

Using L'Hopital's rule and the previous equation, the value of phase delay for $\omega \to 0$ is:

$$\tau_p(\omega)_{|\omega \to 0} = \sqrt{\frac{d\beta^2 / d\omega}{d\omega^2 / \omega}}_{|\omega \to 0} = \sqrt{\tau_{pg}(\omega)_{|\omega \to 0}} \quad (9)$$

$$\tau_p(\omega)_{\omega \to 0} = \frac{(LG + RC)/2}{\sqrt{RG}} = \sqrt{LC}\,\frac{(LG + RC)/2}{\sqrt{RGLC}} \quad (10)$$

Applying $\omega \to 0$ on Eq.(7) and using Eq.(9) it can be shown that the values of phase delay and group delay are equal for $\omega \to 0$:

$$\tau_g(\omega)_{\omega \to 0} = \frac{(LG + RC)/2}{\sqrt{RG}} = \sqrt{LC}\,\frac{(LG + RC)/2}{\sqrt{RGLC}} \quad (11)$$

Similarly, the values of phase and group delay for $\omega \to \infty$, can be found as:

$$\tau_p(\omega)_{|\omega \to \infty} = \tau_g(\omega)_{|\omega \to \infty} = \sqrt{LC} \quad (12)$$

The values of phase constant and characteristic impedance for these two end frequencies are easily obtained from Eqs.(3) and (4), respectively, and they are:

$$\beta(\omega) = \begin{cases} 0, & \omega \to 0 \\ \infty, & \omega \to \infty \end{cases}, \; Z_c(\omega) = \begin{cases} \sqrt{R/G}, & \omega \to 0 \\ \sqrt{L/C}, & \omega \to \infty \end{cases} \quad (13)$$

To determine the limit values of attenuation, the following relation between attenuation and phase constant is used:

$$\alpha(\omega) = \frac{\omega(LG + RC)}{2\beta(\omega)} = \frac{(LG + RC)/2}{\beta(\omega)/\omega} \quad (14)$$

Having in mind that denominator in last equation represents phase delay whose values for two end frequencies are already determined by Eqs.(10) and (12), the values for attenuation constant for $\omega \to 0$ and $\omega \to \infty$ are:

$$\alpha(\omega) = \begin{cases} \sqrt{RG}, & \omega \to 0 \\ \sqrt{RG}\,\dfrac{(LG + RC)/2}{\sqrt{RGLC}}, & \omega \to \infty \end{cases} \quad (15)$$

III. HEAVISIDE CRITERION

The condition of minimum losses is obtaned from $d\alpha^2/dL = 0$ which, using Eq.(2), can be expressed as [4]:

$$\sqrt{\left(\frac{G^2 + \omega^2 C^2}{R^2 + \omega^2 L^2}\right)}\omega^2 L - \omega^2 C = 0 \quad (16)$$

From the last equation, the well-known Heaviside criterion is yielded: $LG=RC$. When this relation is fulfilled, attenuation along a line and its characteristic impedance are independent of the frequency while a phase characteristic linearly depends on frequency. Eqs.(2), (3) and (4) are simplified:

$$\alpha(\omega) = \sqrt{RG} \quad (17)$$

$$\beta(\omega) = \omega\sqrt{LC} \quad (18)$$

$$Z_c(\omega) = \sqrt{R/G} \quad (19)$$

The transmission line with such primary per-unit length parameters, satisfying Heaviside criterion, imposes an equal phase and group delay for all components in signal spectrum which means that any signal will be propagated along the line without distortion:

$$\tau_p(\omega) = \tau_g(\omega) = \sqrt{LC} \quad (20)$$

IV. HEAVISIDE CRITERION FULFILLMENT FACTOR

For the transmission line with primary per-unit length parameters R, L, G and C, the required inductance for fuffillment factor of Heaviside criterion is obtaned as:

$$L_H = \frac{RC}{G} \quad (21)$$

We note the ration of per-unit length inductance of the line and required Heaviside's per-unit length inductance as $l=L/L_H$.

The values of characteristic propagation functions for two end frequencies $\omega\to 0$ and $\omega\to\infty$ as a function of factor l as well as their values for l=1 are given in Table 1.

Propagation functions	$\omega\to 0$	$\omega\to\infty$	*Heaviside* condition (l=1)
$\beta(\omega)$	0	∞	$\beta_H = \omega\sqrt{L_H C}$
$\alpha(\omega)/\alpha_H$	1	$\dfrac{(l+1)/2}{\sqrt{l}}$	$\alpha_H = \sqrt{RG}$
$\lvert Z_c(\omega)\rvert / Z_{cH}$	1	$\sqrt{l}$	$Z_{cH} = \sqrt{R/G}$
$\tau_p(\omega)/\tau_{pH}$	$(l+1)/2$	$\sqrt{l}$	$\tau_{pH} = \sqrt{L_H C}$
$\tau_g(\omega)/\tau_{gH}$	$(l+1)/2$	$\sqrt{l}$	$\tau_{gH} = \sqrt{L_H C}$

Table 1 *Values of characteristic propagation functions for $\omega\to 0$, $\omega\to\infty$ and Heaviside criterion*

In order to graphically illustrate the areas of changes of characteristic propagation functions with frequency, new functions have been introduced and shown in Fig.1:

$$f_a(l) = \frac{l+1}{2},\quad f_g(l) = \sqrt{l}\ ,\quad f_k(l) = \frac{f_a(l)}{f_g(l)} = f_k(1/l) \quad (22)$$

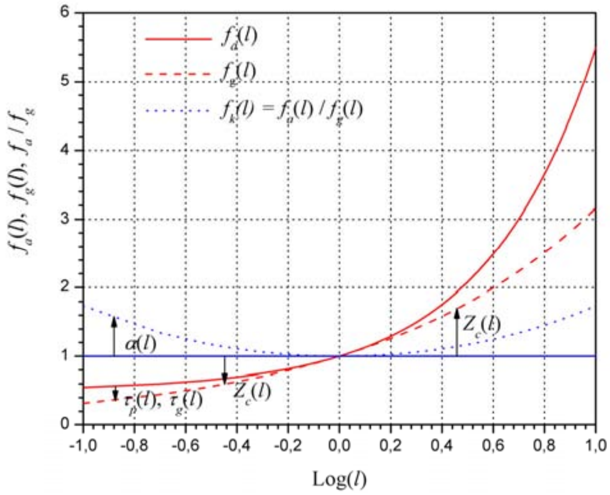

Fig.1 *Areas of changes of characteristic propagation functions with frequency versus factor l*

The lines used in telecommunications have a small per-unit length inductance and as a result, factor l is, in reality, always smaller than 1 (even with inserted Pupin coils) but for the purpose of theoretical analysis, the values of l bigger than 1 are taken into account as well. From the Fig.1 it can be noticed that the increase of factor l narrows the area of possible changes of attenuation and characteristic impedance as well as phase and group delay with frequency. This behaviour is valid until l equal to 1 and then there is a reversed proces with the further increase of l. In addition, around l=1, the changes of these parameters with frequency can be neglected. From the Table 1 and Fig.1 the following relations which identify the areas of changes of attenuation constant, characteristic impedance, phase and group delay with factor l are:

$$1 \le \alpha(\omega)/\alpha_H \le f_k(l) \quad (23)$$

$$f_g(l) \le \lvert Z_c(\omega)\rvert / Z_{cH} \le 1,\ l \le 1$$
$$1 \le \lvert Z_c(\omega)\rvert / Z_{cH} \le f_g(l),\ l \ge 1 \quad (24)$$

$$f_a(l) \ge \tau_{p(g)}(\omega)/\tau_{p(g)H} \ge f_g(l) \quad (24)$$

Dependance of attenuation constant, characteristic impedance, phase and group delay, normalized with their value at fulfilled Heaviside criterion, as well as the value of phase constant for different values of factor l are shown in Figs.2,3,4,5 and 6, respectively. The considered frequency range is up to 4 KHz because it is suitable to the baseband telephone signal transmission. Besides, at these relatively low frequencies, the variation of primary per-unit length parameters with frequency can be neglected. Primary parameters used for calculation are those used for La Manche channel cable [5]: R=14.2 Ω/mile, C=138 nF/mile and G=24 μS/mile. Inductance required for fulfillment of Heaviside criterion is obtained from Eq.(21) as L_H=0.08165 H/mile. The value of factor l is changed by increasing per-unit length inductance of line.

From the Fig.2 it can be seen that normalized attenuation curves, independantly from factor l, start from 1 or α_H, and then monotonically increase approaching asymptotically, with frequency increase, to the lines determined with function $f_k(l)$. This behaviour is the same for factor l and 1/l, because $f_k(l)$= $f_k(1/l)$, (curve $\alpha(l)$ is symmetrical in regard to l=1, see Fig.1).

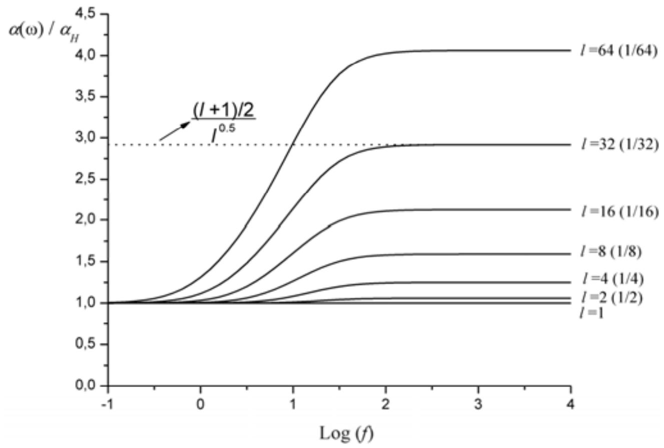

Fig.2 *Normalized function $\alpha(\omega)$ for different values of factor l*

Characteristic impedance curves show a different behaviour for factor l and 1/l (Fig.3). In both cases, these curves start from 1 or Z_{cH} , and with frequency, they are approaching asymptotically to the lines $f_g(l)$. However, in the first region (l<1), curves of characteristic impedance monotonically decrease, while in the second region (l>1) monotonically increase.

Curves of phase and group delay (Fig.4 and 5), for given l, start from the points defined by function $f_a(l)$ and, with frequency increase, they are approaching to the lines whose location is determined by function $f_g(l)$. In the frequency range up to 100 Hz, the changes of phase and group delay with frequency are the most significant. For l>1 delays are greater than the values corresponding to Heaviside criterion, $\tau_{pH} = \tau_{gH}$, given in Table 1, and for l<1 inserted delays are smaller. In addition, for the same value of factor l, phase delay and group delay start from the same point and finish at the same point.

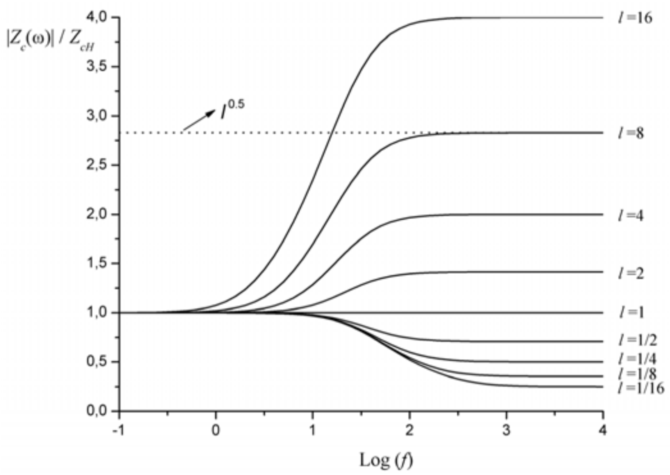

Fig.3 *Normalized function $Z_c(\omega)$ for different values of factor l*

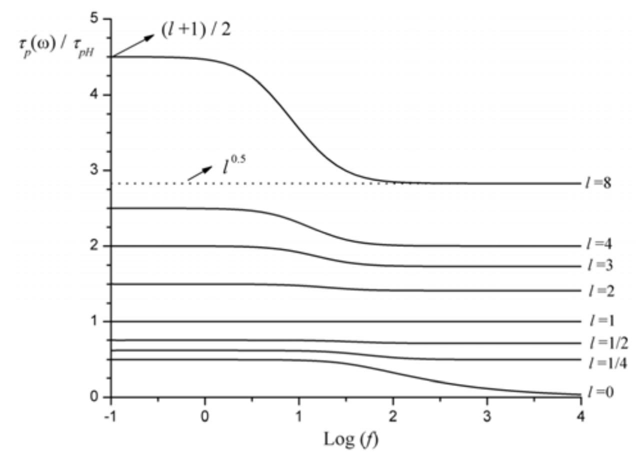

Fig.4 *Normalized function $\tau_p(\omega)$ for different values of factor l*

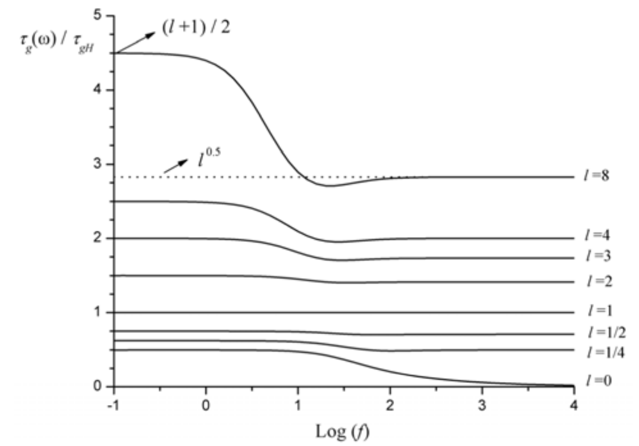

Fig.5 *Normalized function $\tau_g(\omega)$ for different values of factor l*

In Fig.6 the values of phase constant, calculated by using Eq.(3), are shown for different values of factor l. It can be noticed that the curve representing dependance of phase constant with frequency is linearized with increase of l. For l=1 the value of phase constant corresponds to its value at fulfilled Heaviside criterion, β_H, given in Table 1. In that case, signal is propagated along the line without dispersion while phase velocity and group velocity are equal at all frequencies.

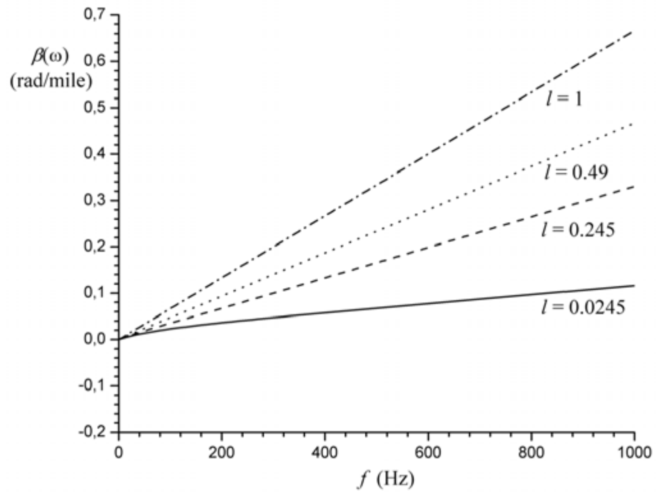

Fig.6 *Function $\beta(\omega)$ for different values of factor l*

V. CONCLUSION

In this paper, a behavior of characteristic functions, definining propagation along a lossy TEM transmision line, as a function of per-unit length inductance increase is analysed. For an analysis purpose, a new parameter - Heaviside criterion fulfillment factor is introduced. Limit values of considered functions are expressed through the proposed factor. Presented results allow for, for an example, an estimation for how much per-unit length inductance should be increased (by inserting Pupin coils) to keep changes of attenuation, phase and group delay in acceptable limits from the signal propagation dispersion point of view. Future research will be based on a full wave matrix analysis of pupinized transmission line as a function of number of equally inserted discrete coils per wavelength corresponding to the maximum frequency of interest.

REFERENCES

[1] L. Cohen, *Heaviside's Electrical Circuit Theory*, McGraw Hill Book Co., London, 1928.

[2] M. Pupin, *Art of Reducing Attenuation of Electrical Waves and Apparatus Therefore*, patent No. 652 230, June 19, 1900, application filed December 14, 1899.

[3] P. Miljanić, *Theory of Periodical Electric Currents Propagation*, Edition of Serbian Academy of Sciences and Arts, Beograd, 2004, in Serbian.

[4] P.F. Combes, *Microwave Transmission for Telecommunications*, John Wiley & Sons, 1991.

[5] A. Marinčić, R. Biljić, "Heaviside Criterion and Cable Bandwidth", *XII Telecommunications Simposium TELFOR 2004*, Beograd, 23-25 November 2004, in Serbian.

[6] A. Marinčić, "Analysis of Transmission Line with Periodical Inductance Loading", *Microwave Review*, Vol.10. No.2, 2004, pp.43-48.