Convergence Analysis of an LMS Adaptive Low Sensitivity Biquadratic Digital Filter Section

Maria Nenova[1], Georgi Iliev[2], Georgi Stoyanov[3]

Abstract – **This paper presents a mean convergence analysis of a low sensitivity biquadratic filter section transformed into adaptive by using an LMS algorithm. The structure allows independent tuning of the central frequency and the bandwidth of the bandpass/bandstop realizations. The mean convergence analysis of the adaptation process is then presented. All results received for theoretical bounds are also verified experimentally.**

Keywords –**notch IIR structure, adaptive LMS algorithm, convergence analysis**

I. Introduction

Over the years, a variety of IIR structures for adaptive digital filters have been investigated and implemented [1], [2], [3]. Many of them are second-order sections [2], but they are not useful for a realization in the low frequency band, because of their high sensitivity. A biquadratic section (called BQ3) is investigated thoroughly in [4], [5] and in [4] BQ3 is improved so that the section is very efficient for applications in the low frequency band, resulting from its considerably low sensitivity of the poles near z=1. An important feature when designing second-order section is that they should employ a canonic number of multipliers (two in BQ3) in contrast to the structures investigated in [6]. This second-order section is also implemented here because it allows independent tuning of the central frequency and the bandwidth of bandbasss/bandstop realizations.

Initially, the adaptive algorithms were of Gauss-Newton's type or its simplifications [1], [6], [7]. Most of them include modifications of the gradient oriented algorithms as in [3]. Moreover in [7] it is shown that a significant number of those algorithms have a high computational complexity and slow convergence. In [8] a modification of the least-squares algorithm is proposed, avoiding matrix inversion and having very fast convergence.

In this paper the BQ3 section is first transformed into adaptive by incorporating an LMS adaptive algorithm. This algorithm is one of the widely used in the adaptive signal processing,because of its low computational complexity.

Then mean convergence analysis, similar to the one in Ref.[1] is performed.

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Finally some experiments are conducted to verify the theoretical results.

II. Design Method Description

The biquadratic section BQ3, given in Fig.1 was proposed in [4],[5], it is realizing (at different outputs) bandpass/bandstop transfer functions and it was shown that it has extremely low sensitivities for pole near z=1.

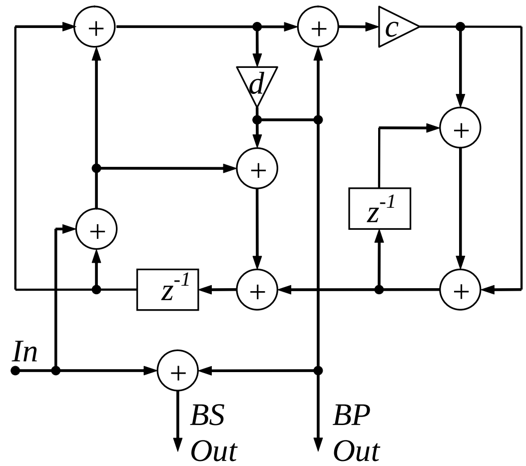

Fig. 1 Biquadratic section BQ3 with its BP and BS outputs

The transfer functions at the BP and BS output of the structure are [5]:

$$H_{BP}(z) = \frac{d(1-z^{-2})}{1+(-2+4c+2d-4cd)z^{-1}+(1-2d)z^{-2}}, \quad (1)$$

$$H_{BS}(z) = \frac{(1-d)[1-2(1-2c)z^{-1}+z^{-2}]}{1+(-2+4c+2d-4cd)z^{-1}+(1-2d)z^{-2}}. \quad (2)$$

Their bandwidth BW:

$$BW = \cos^{-1}\frac{1-2d}{1-2d+2d^2}, \quad (3)$$

is depending only on d, so it can be controlled and tuned independently.

The central frequency depends generally on the coefficients c and d, but once the BW is tuned and d – fixed, this frequency can easily be tuned by changing c.

When the BS output is used:

$$\theta_z = \cos^{-1}(1-2c) \quad (4)$$

III. LMS Adaptive Algorithm

Usually when a BS filter is used for adaptive tracking of a narrow band signal. The BW is fixed to some given value and thus only the center frequency becomes a subject of adaptation. It is seen from (4) that in BQ3 it could easily be performed by changing *c*. If the adaptation is done according to the Least Mean Squares algorithm, the updated value of *c* will be calculated as:

$$c(n+1) = c(n) - \mu e(n) g(n). \quad (5)$$

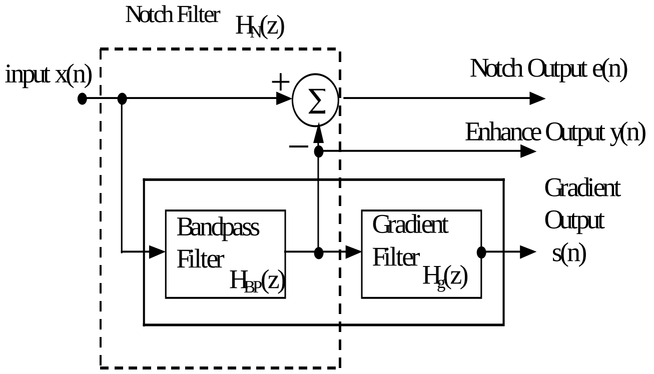

Fig.2 An adaptive IIR notch filter

The adaptive notch filter configuration and the whole structure are shown in Fig.2.

In equation (5) $g(n)$ is the gradient function of $e(n)$ with respect to the coefficient *c*. If $e(i)$ represents the error signal at the BS output on Fig.1 and $x(i)$ is the input signal, we can easily obtain from Eq.(2) the following difference equation:

$$\begin{aligned} e(i) = (1-d)x(i) + (-2+4c(i)+2d-4dc(i))x(i-1) \\ +(1-d)x(i-2) - (-2+4c(i)+2d-4dc(i)) \quad (6) \\ e(i-1) - (1-2d)e(i-2). \end{aligned}$$

Hence the signal $g(n)$ in function of *c* is:

$$g(i) = \frac{\partial e(i)}{\partial c(i)} = 4(1-d)[x(i-1) - e(i-1)]. \quad (7)$$

IV. Mean Convergence

Let's assume that the input signal consists of an unknown sinusoid and a noise. The noise is white Gaussian with zero-mean

$$x(i) = A\cos(\omega_0 i + \theta) + v(i)\,, \quad (8)$$

where *A* represents the sinusoid amplitude, θ is the initial phase, ω_0 is the angular frequency and $v(i)$ is the noise. The frequency of the input sinusoid is estimated by searching the optimum coefficient *c* for which the adaptation error is minimal.

Investigations are based on the assumption of the local linearization [9] of the gradient function about the point where the function is stationary, the error surface is flat in areas far from the minimum and is steepest near the stationary point It gives us a possibility to constrain stability bounds for μ around the area of the optimal solution.

The signals $e(i)$ and $g(i)$can be expressed not only with a difference equations, but also as:

$$e(i) = \left|H_N(e^{j\omega_0})\right| A\cos(\omega_0 i + \theta + \varphi_N(\omega_0)) + v_1(i), \quad (9)$$

$$g(i) = \left|H_g(e^{j\omega_0})\right| A\cos(\omega_0 i + \theta + \varphi_g(\omega_0)) + v_2(i). \quad (10)$$

Here $H_g(e^{j\omega_0})$ is the gradient transfer function and $H_N(e^{j\omega_0})$ is the notch transfer function.

Because of the linearization about the stationary point it can be presumed that near this point the notch transfer function is a linear function of *c*. Therefore the gradient function is constant

$$H_{BP}(e^{j\omega_0}) = \frac{d(1-e^{-2j\omega_0})}{D(e^{j\omega_0})} \approx 1. \quad (11)$$

Hence the denominator $D(z)$ can be approximated to:

$$D(e^{j\omega_0}) \approx d(1-e^{-2j\omega_0}). \quad (12)$$

Substituting (11) and (12) into the expression for the bandpass transfer function and the gradient transfer functions result in:

$$\begin{aligned} H_{BS}(e^{j\omega_0}) = \frac{4(1-d)e^{-j\omega_0}c^*}{D(e^{j\omega_0})} = \frac{4(1-d)e^{-j\omega_0}c^*}{d(1-e^{-2j\omega_0})} = \\ = \frac{2(1-d)c^*}{d\sin\omega_0}, \end{aligned} \quad (13)$$

$$\begin{aligned} H_g(e^{j\omega_0}) = \frac{4(1-d)e^{-j\omega_0}}{D(e^{j\omega_0})} = \frac{4(1-d)e^{-j\omega_0}}{d(1-e^{-2j\omega_0})} = \\ = \frac{2(1-d)}{d\sin\omega_0}. \end{aligned} \quad (14)$$

For simplification we use the following notation for the common term of (13) and (14):

$$B = \frac{2(1-d)}{d\sin\omega_0}. \quad (15)$$

The exact value of the adaptive coefficient *c* is:

$$c = \delta + c^*, \quad (16)$$

where δ represents the deviation from the optimal solution c^*. For further simplification we introduce:

$$\gamma_i = \omega_0 i + \theta + \varphi(\omega_0). \quad (17)$$

into (9) and (10) and, having in mind (11) – (14)finally get:

$$e(i) \approx AB(c^* + \delta)\cos(\gamma_i) + v_1(i), \quad (18)$$

$$g(i) \approx AB\cos(\gamma_i) + v_2(i). \quad (19)$$

V. Mean Convergence Analysis

Using the expression for the adaptive algorithm in (5) and (16) $\delta(i) = c(i) - c^*$ we derive the update of the adaptive coefficient c :

$$\begin{aligned}\delta(i+1) \approx \delta(i) - \mu[A^2B^2(c^* + \delta)\cos^2(\gamma_i) + \\ ABc^* \cos(\gamma_i)v_2(i) + AB\cos(\gamma_i) \\ v_1(i) + v_1(i)v_2(i)].\end{aligned} \quad (20)$$

In order to perform the mean analysis we introduce some additional, simplifications similar to those in [1]. The sinusoid phase θ is assumed to be uniformly distributed in [-π, π] and the white Gaussian noise has a variance σ^2_n. We have assumed that there is approximately no correlation between $cos(\gamma_i)$ and $v_1(i)$, as well as between $cos(\gamma_i)$ and $v_2(i)$. Consequently, the correlation between $cos(\gamma_i)$ and $\delta(i)$ and between $v_1(i)$ and $v_2(i)$ is zero, because they are uncorrelated.

Using this assumptions for the mathematical expectation of $\delta(i)$ the expression (20) becomes:

$$E[\delta(i+1)] \approx E[\delta(i)] - \mu E\begin{bmatrix} A^2B^2\delta(i)\cos^2\gamma_i + \\ + A^2B^2c^*\cos^2\gamma_i \end{bmatrix}. \quad (21)$$

If we neglect the term not containing $\delta(i)$ we finally get:

$$E[\delta(i+1)] \approx E[\delta(i)]\left(1 - \mu\frac{A^2B^2}{2}\right). \quad (22)$$

If the algorithm is to be stable it is required that :

$$\left|1 - \mu\frac{A^2B^2}{2}\right| < 1. \quad (23)$$

We have chosen a stricter criterion for μ than what is taken in[1], as seen from (21). Solving the expression (23) with respect to μ we will obtain:

$$0 < \mu < \frac{d^2\sin^2\omega_0}{A^2(1-d)^2} \equiv \mu_{\max}. \quad (24)$$

The step size depends on the sinusoid amplitude and on d which is responsible for the pole radius.

This expression shows how to control the upper bound of the step size μ and the dependence of μ on the sinusoid amplitude and the pole radius expressed by:

$$d = \frac{1-r^2}{2}. \quad (25)$$

VI. Simulation Results

The simulation are made for the signal-to-noise ratio SNR=20dB and SNR=10dB. For SNR=20dB, the amplitude of the input sinusoid is unity and the noise variance σ_n^2 is 0.1. The step size μ and coefficient d are varied in order to investigate the adaptation process for different values. Experiments are performed with f_s=8 kHz.

The adaptation process of the LMS algorithm in (5) to sinusoid with frequency 200Hz, for different bandwidths is illustrated in Fig. 3.

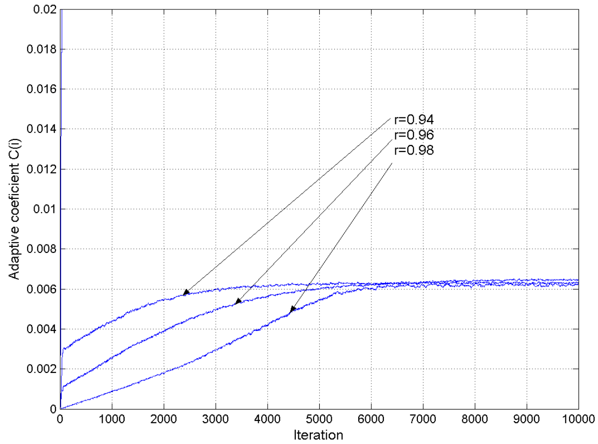

Fig. 3 Adaptation with different bandwidths for frequency of 200Hz

From the learning curves it can be concluded that the greater the pole radius/BW is, the faster the adaptation speed is.

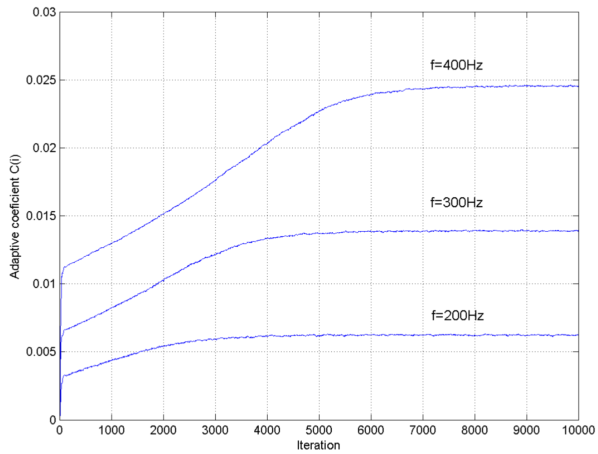

Fig.4 Investigation of tracking of different frequencies with SNR=10dB

It is seen that the best result is achieved for the lowest frequency, which is due to the implemented structure features.

The theoretical bounds for the step size μ for different pole radii are compared graphically with the experimental bounds for a sinusoidal signal with a frequency of 200Hz in Fig.5:

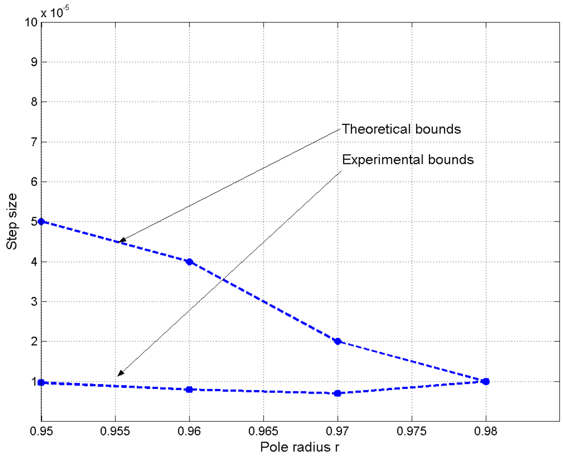

Fig.5 Theoretical and experimental stability bounds for f=200Hz

It is also seen that the experimentally obtained bound is staying constant for all values of r.

Fig.6 presents the analytical and theoretical bounds for μ for two different SNR's.

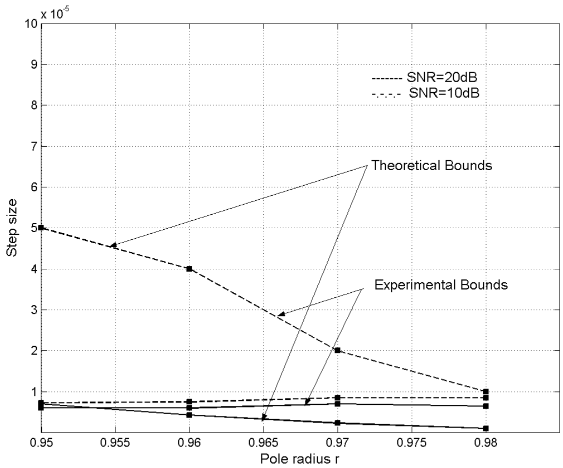

Fig.6 Investigation of theoretical and experimental bounds for different SNRs

We can conclude from this figure that the step sizes of μ are approximately equal for the two values of SNR.

VII. Conclusions

A new biquadratc filter section with independent tunning of the central frequency and the bandwidth was turned to adaptive with an LMS algorithm for adaptation of the central frequency.

Then the mean convergence of the notch filter so obtained was investigated and the bounds for the adaptation step size have been defined. A number of simplifications have been performed and the theoretical results for the step size bounds have been confirmed. A mean square convergence analysis still remains to be done.

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