A Method for Synthesis of Generalized Barker Codes

Borislav Y. Bedzhev[1], Zhaneta N. Tasheva[2] and Borislav P. Stoyanov[3]

Abstract - **Signals, which auto-correlation function (*ACF*) has as small as possible side lobes, are preferred in communication applications. From historical point of view, the Barker codes are the first class of signals with this property. Due to their positive features the Barker codes have been studied intensively since their introduction in 1953. Despite of the taken efforts, there a lot of open problems exist still. With regard our paper suggests a method for synthesis of so-named six-phase or sextic generalized Barker codes (*SGBC*). Some aspects of practical applying of the suggested method are discussed. An unknown till now *SGBC* of length *n* = 18 is presented also.**

Keywords- **Communication system signaling, sextic generalized Barker codes.**

I. INTRODUCTION

The communications are in very rapid progress today. As a result, a lot of radio, television and local area wireless services are working together at every place and at very time. This situation leads to undesirable interferences of different signals and decreases the performance quality of the communication services. As known, the most effective method for precluding of multiple access interferences (*MAI*) is the usage of signals, which auto-correlation function (*ACF*) has as small as possible side lobes. It is necessary to emphasize that radio signals with this property are very important for some particular applications such as radars, time-synchronous networks and radio navigation (including global positioning) systems.

From historical point of view, the so-named Barker sequences or codes are the first class of signals with above type of *ACF* [1], [2]. As known, the Barker codes are radio signals, which are generated only by means of phase modulation. More specifically, an arbitrary complex phase modulated signal consists of *n* elementary pulses (waveforms) with duration τ which complex envelopes are:

$$\xi(j-1) = U_j \exp(i.\theta_j);\ j = 0,1,...n-1, \qquad (1)$$

[1]Borislav Y. Bedzhev is with the Faculty of Artillery, Air Defense and Communication Information System, National Military University, 1 Karel Shkorpil Str., 9713 Shoumen, Bulgaria, E-mail: bedzhev@mail.pv-ma.bg

[2]Zhaneta N. Tasheva is with the Faculty of Artillery, Air Defense and Communication Information System, National Military University, 1 Karel Shkorpil Str., 9713 Shoumen, Bulgaria, E-mail: tashevi86@yahoo.com

[3]Borislav P. Stoyanov is with the Faculty of Mathematics, Informatics and Economics, Shumen University, 1 Universitetska Str., 9700 Shoumen, Bulgaria, E-mail: bpstoyanov@abv.bg

where U_j and θ_j are the amplitude and the phase of j-th elementary pulse respectively. In this case the autocorrelation properties of a phase manipulated (*PM*) signal can be comprehensibly described by means of the aperiodic *ACF* of the sequence $\{\xi(j)\}_{j=0}^{n-1} = \xi(0),\xi(1),...\xi(j),...\xi(n-1)$, evaluated only in time-points $k\tau$:

$$R_{\xi\xi}(k) = \begin{cases} \sum_{j=0}^{n-1-|k|} \xi(j).\xi^*(j+|k|), & -(n-1) \le k \le 0, \\ \sum_{j=0}^{n-1-k} \xi^*(j).\xi(j+k), & 0 \le k \le (n-1). \end{cases} \qquad (2)$$

Here symbol "*" means "complex conjugation.

By definition, the Barker codes are the *PM* signals with the property:

$$|R(k)| \le 1,\ \ k \ne 0\,. \qquad (3)$$

Here the subscripts of *R* are omitted and this notation will be used for convenience henceforth.

In order to obtain maximal energy effectiveness of the transmitter and to simplify the complex processes in the communication devices the following limitations are imposed in the practice very often:

$$U_j = U_0 = const,\ \theta_j \in \{[(2\pi.l)/m];\ l = 0 \div m-1\}. \qquad (4)$$

The *PM* signals, satisfying (3), are named "*uniform*" and in the rest part of this paper our attention shall be focused on this type of *PM* signals.

Due to their positive features the Barker codes have been studied intensively since their invention in 1953 [3], [4], [5], [6], [7], [8], [9], [10]. Despite of the taken efforts, there a lot of open problems exist still [9]. For instance, originally the Barker codes were introduced as binary manipulated signals (i.e. $m = 2$ in (4)) but it has been found that they do not exist if *n* is an odd integer greater than 13 and may not exist if *n* is an even integer greater than 4. The later conjecture is not proved still (see [9], [10]).

With regard to the all above cited, our paper aims to suggest a method for synthesis of so-named six-phase or sextic generalized Barker codes (i.e. in (4)) [6], [9].

The paper is organized as follows. First, the basics properties of sextic generalized Barker codes (SGBC) are recalled. After then, a method for synthesis of SGBC is described. Some aspects of practical applying of the suggested method are discussed also. Finally, an unknown till now SGBC of length is presented.

II. BASICS OF SEXTIC GENERALIZED BARKER CODES

In this section of our report we shall prove some necessary conditions which must satisfy the aperiodic *ACF* of an arbitrary *SGBC*. They will be used in order to reduce the possible variants in the process of *SGBC* synthesis.

At the beginning we shall recall that our attention will be focused on the *SGBC* [6], [9]. This means that $m = 6$ in (4) and according to Eq. (1) the complex envelopes $\xi(j-1),\ j = 0,1,...n-1,$ of the elementary pulses (waveforms) with duration τ, forming a *SGBC*, belong to the set $\{\pm U_0,\ \pm \omega U_0,\ \pm \omega^2 U_0\}$. Here $\{\pm 1,\ \pm \omega,\ \pm \omega^2\}$ are the sixth roots of unity:

$$e^{\frac{2\pi}{6}i} = \frac{1}{2} + i\frac{\sqrt{3}}{2} = -\omega^2; e^{\frac{2\pi}{6}2i} = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = \omega; e^{\frac{2\pi}{6}3i} = -1;$$
$$e^{\frac{2\pi}{6}4i} = -\frac{1}{2} - i\frac{\sqrt{3}}{2} = \omega^2; e^{\frac{2\pi}{6}5i} = \frac{1}{2} - i\frac{\sqrt{3}}{2} = -\omega; e^{\frac{2\pi}{6}6i} = 1. \quad (5)$$

Without losing of generality, it can be assumed $U_0 = 1[V]$ in Eq. (5). Now the following Proposition 1 will be stated.

Proposition 1: The side lobes of the aperiodic *ACF* of a *SGBC* can take only the values:

$$R(k) = 0,\quad \pm 1,\quad \pm \omega,\quad \pm \omega^2;\ k \neq 0. \quad (6)$$

Proof: As known, the complex numbers $\{\pm 1,\ \pm \omega,\ \pm \omega^2\}$ form a multiplicative group denoted as *C(6)* often. From this fact one can observe that the sum $R(k) = \sum_{j=0}^{n-1-k} \xi(j)\xi^*(j+k)$ comprises *n-1-k* terms, which belong to the set $\{\pm 1,\ \pm \omega,\ \pm \omega^2\}$. Hence:

$$R(k) = \alpha + \beta\omega + \gamma\omega^2, \quad (7)$$

where α, β и γ are integers. From Eq. (7) the magnitude of a *SGBC ACF* side lobe can be presented in the form:

$$|R(k)|^2 = (\alpha + \beta.\omega + \gamma.\omega^2)(\alpha + \beta.\omega + \gamma.\omega^2)^* =$$
$$= (\alpha + \beta.\omega + \gamma.\omega^2)(\alpha + \beta.\omega^2 + \gamma.\omega), \quad (8)$$

because $\omega^* = \omega^2$ и $(\omega^2)^* = \omega$ (see Eq. (5)). After opening of parentheses in Eq. (8), the result is:

$$|R(k)|^2 = \alpha^2 + \beta^2 + \gamma^2 + (\alpha.\gamma + \alpha.\beta + \gamma.\beta).\omega +$$
$$+ (\alpha.\beta + \beta.\gamma + \alpha.\gamma).\omega^2 = \alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \alpha\gamma - \beta\gamma. \quad (9)$$

Here it is taking into account that $1 + \omega + \omega^2 = 0$, $\omega^3 = 1$ and consequently:

$$(\alpha.\gamma + \alpha.\beta + \gamma.\beta).\omega + (\alpha.\beta + \beta.\gamma + \alpha.\gamma).\omega^2 =$$
$$(\alpha\beta + \beta\gamma + \alpha\gamma)(\omega + \omega^2) = -(\alpha\beta + \beta\gamma + \alpha\gamma). \quad (10)$$

Now the Eq. (9) can be presented in the form:

$$|R(k)|^2 = \frac{1}{2}\left[2\alpha^2 + 2\beta^2 + 2\gamma^2 - 2\alpha\beta - 2\alpha\gamma - 2\beta\gamma\right] =$$
$$= \frac{1}{2}\left[(\alpha^2 - 2\alpha\beta + \beta^2) + (\alpha^2 - 2\alpha\gamma + \gamma^2) + (\beta^2 - 2\beta\gamma + \gamma^2)\right] =$$
$$= \frac{1}{2}\left[(\alpha-\beta)^2 + (\alpha-\gamma)^2 + (\beta-\gamma)^2\right]. \quad (11)$$

After substituting with Eq. (11) in (3) one can obtain:

$$\frac{1}{2}\left[(\alpha-\beta)^2 + (\alpha-\gamma)^2 + (\beta-\gamma)^2\right] \leq 1, \quad (12)$$

and hence:

$$(\alpha-\beta)^2 + (\alpha-\gamma)^2 + (\beta-\gamma)^2 \leq 2. \quad (13)$$

It is straightforward that possible solutions of Eq. (13) are:

1) $\alpha = \beta = \gamma$;
2) $\beta = \gamma;\quad \alpha = \beta \pm 1 = \gamma \pm 1$;
3) $\alpha = \beta;\quad \gamma = \alpha \pm 1 = \beta \pm 1$;
4) $\alpha = \gamma;\quad \beta = \alpha \pm 1 = \gamma \pm 1.$

(14)

From Eqs. (14) it is apparent that the side lobes of the aperiodic *ACF* of a *SGBC* can take only the values:

1) $R(k) = \alpha + \alpha.\omega + \alpha.\omega^2 = \alpha(1 + \omega + \omega^2) = 0$;
2) $R(k) = (\beta \pm 1) + \beta.\omega + \beta.\omega^2 = \beta(1 + \omega + \omega^2) \pm 1 = \pm 1$;
3) $R(k) = \alpha + \alpha.\omega + (\alpha \pm 1).\omega^2 = \alpha(1 + \omega + \omega^2) \pm \omega^2 = \pm \omega^2$
4) $R(k) = \alpha + (\alpha \pm 1).\omega + \alpha.\omega^2 = \alpha(1 + \omega + \omega^2) \pm \omega = \pm \omega$. (15)

The Eqs. (15) complete the proof of Proposition 1.

III. METHOD FOR SYNTHESIS OF SEXTIC GENERALIZED BARKER CODES

All present available methods for signal synthesis seem to contain an element of trial and error [3], [4], [5], [6], [7], [8], [9], [10]. Consequently, the above proved Proposition 1 allows a significant reducing of computational complexity of programs, which are used for *SGBC* synthesis. This will be explained in more details in this section of our paper.

Our method for synthesis of *SGBC* comprises approximately $\lceil \log_2 n \rceil$ successive steps.

First step begins with the following polynomial presentation of the *ACF* of an arbitrary *PM* signal:

$$P(x) = F(x).F^*(x^{-1}) =$$
$$= R(-(n-1)).x^{-(n-1)} + R(-(n-2)).x^{-(n-2)} + ... + R(0) + ... +$$
$$+ R(n-2).x^{n-2} + R(n-1).x^{n-1} = \sum_{k=-(n-1)}^{n-1} R(k).x^k \ . \qquad (16)$$

Here:

$$F(x) = \xi(n-1).x^{(n-1)} + ... + \xi(1).x + \xi(0) \qquad (17)$$

is the polynomial, corresponding to the sequence $\{\xi(j)\}_{j=0}^{n-1}$ of complex envelopes of elementary pulses (see Eq. (1)), $R(k)$ are the *ACF* lobes, determined by Eq. (2), and $F^*(x^{-1})$ is the so - named reciprocal polynomial:

$$F^*(x^{-1}) = \xi^*(n-1).x^{-(n-1)} +$$
$$+ \xi^*(n-2).x^{-(n-2)} + ... + \xi^*(1).x^{-1} + \xi(0) \ . \qquad (18)$$

The main idea of our method for synthesis of SGBC is the computational complexity to be reduced by factoring of the polynomial $P(x)$**.** In order to realize this idea the Eq. (16) is examined $\bmod(x-1)$, $\bmod(x+1)$, $\bmod(x-\omega)$ and $\bmod(x+\omega)$ during the first step of our method. For instance:

$$P(x) = F(x).F^*(x^{-1}) \ \bmod(x-1) \ , \qquad (19)$$

is equivalent to the substitution $x = 1$ in Eq. (16). After this substitution and taking into account Eqs. (7) and (11), it is apparent that:

$$\sum_{k=-(n-1)}^{n-1} R(k) = \left[\sum_{k=0}^{n-1} \xi(k)\right]\left[\sum_{k=0}^{n-1} \xi^*(k)\right] =$$
$$= \left(\alpha_0 + \beta_0.\omega + \gamma_0.\omega^2\right)\left(\alpha_0^* + \beta_0^*.\omega^{-1} + \gamma_0^*.\omega^{-2}\right) =$$
$$= \left(\alpha_0 + \beta_0.\omega + \gamma_0.\omega^2\right)\left(\alpha_0^* + \beta_0^*.\omega^2 + \gamma_0^*.\omega\right) =$$
$$= \left(\alpha_0 + \beta_0.\omega + \gamma_0.\omega^2\right)\left(\alpha_0 + \beta_0.\omega + \gamma_0.\omega^2\right)^* =$$
$$= \frac{1}{2}\left[(\alpha_0 - \beta_0)^2 + (\alpha_0 - \gamma_0)^2 + (\beta_0 - \gamma_0)^2\right] \qquad (20)$$

Here α_0, β_0 и γ_0 are integers and consequently, the Eq. (20) proves the following Proposition 2.

Proposition 2: The double sum of the *ACF* lobes of a *SGBC* must be a sum of three exact quadrates.

In this way one can find that:

$$\sum_{k=-(n-1)}^{n-1} R(k)(-1)^k = \frac{1}{2}\left[(\alpha_1 - \beta_1)^2 + (\alpha_1 - \gamma_1)^2 + (\beta_1 - \gamma_1)^2\right], (21)$$

$$\sum_{k=-(n-1)}^{n-1} R(k)\omega^k = \frac{1}{2}\left[(\alpha_2 - \beta_2)^2 + (\alpha_2 - \gamma_2)^2 + (\beta_2 - \gamma_2)^2\right], (22)$$

$$\sum_{k=-(n-1)}^{n-1} R(k)(-\omega)^k =$$
$$= \frac{1}{2}\left[(\alpha_3 - \beta_3)^2 + (\alpha_3 - \gamma_3)^2 + (\beta_3 - \gamma_3)^2\right]. \qquad (23)$$

The number of all possible *ACF* of a *SGBC* is:

$$L(n) = (n-1)^7 \qquad (24),$$

because $R(-k) = R^*(k)$, $R(0) = n$ accordingly to (2) and due to Proposition 1. It ought to emphasize that Eqs. (6), (20)-(23) contain necessary conditions, which allow reducing the number $L(n)$ of possible *ACF* significantly. First step of our methods for *SGBC* synthesis ends with forming of a massive, containing the *ACF*, which have passed all sieves in Eq. (6), (20)-(23).

At the second step of our method every polynomial $P(x) = F(x).F^*(x^{-1})$ (see Eq. (16)), corresponding to a possible *ACF* found in the first step, is examined $\bmod(x^3-1)$. It is utilized that:

$$x^3 - 1 = (x-1)(x-\omega)(x-\omega^2) \ , \qquad (25)$$

$$F(x).F^*(x^{-1}) \quad \bmod(x^3-1) =$$
$$= \left[\sum_{k=0}^{\left\lceil\frac{n-1}{3}\right\rceil} \xi(3k) + \sum_{k=0}^{\left\lceil\frac{n-1}{3}\right\rceil} \xi(3k+1)x + \sum_{k=0}^{\left\lceil\frac{n-1}{3}\right\rceil} \xi(3k+2)x^2\right] \times$$
$$\times \left[\sum_{k=0}^{\left\lceil\frac{n-1}{3}\right\rceil} \xi^*(3k) + \sum_{k=0}^{\left\lceil\frac{n-1}{3}\right\rceil} \xi^*(3k+1)x^{-1} + \sum_{k=0}^{\left\lceil\frac{n-1}{3}\right\rceil} \xi^*(3k+2)x^{-2}\right] =$$
$$= \sum_{k=-\left\lceil\frac{n-1}{3}\right\rceil}^{\left\lceil\frac{n-1}{3}\right\rceil} R(3k) + \sum_{k=0}^{\left\lceil\frac{n-1}{3}\right\rceil}\left[R(-3k-1) + R(3k+1)\right]x +$$
$$+ \sum_{k=0}^{\left\lceil\frac{n-1}{3}\right\rceil}\left[R(-3k-1) + R(3k+2)\right]x^2 \quad \bmod(x^3-1) \ . \qquad (26)$$

The second step of our method can be explained as follows. At the beginning, the possible *ACF* are taken successively from the massive, formed at the end of the first step. After that, the system of following equations:

$$\left[\sum_{k=0}^{n-1} \xi(k)\right]\left[\sum_{k=0}^{n-1} \xi^*(k)\right] = \sum_{k=-(n-1)}^{n-1} R(k)$$
$$\left[\sum_{k=0}^{n-1} \xi(k)\omega^k\right]\left[\sum_{k=0}^{n-1} \xi^*(k)\omega^{-k}\right] = \sum_{k=-(n-1)}^{n-1} R(k)\omega^k$$

$$\left[\sum_{k=0}^{n-1}\xi(k)(-\omega)^{k}\right]\left[\sum_{k=0}^{n-1}\xi^{*}(k)(-\omega)^{-k}\right]=\sum_{k=-(n-1)}^{n-1}R(k)(-\omega)^{k} \quad (27)$$

is solved, which allows the finding of the three partial sums $\sum_{k=0}^{\lceil \frac{n-1}{3} \rceil}\xi(3k)$, $\sum_{k=0}^{\lceil \frac{n-1}{3} \rceil}\xi(3k+1)$, $\sum_{k=0}^{\lceil \frac{n-1}{3} \rceil}\xi(3k+2)$ in the left side of Eq. (26).

The Eq. (16) is examined $\mathrm{mod}\,(x^{6}-1)$ analogously during the third steps of our method. This allows finding the partial sums $\sum_{k=0}^{\lceil \frac{n-1}{6} \rceil}\xi(6k)$, $\sum_{k=0}^{\lceil \frac{n-1}{6} \rceil}\xi(6k+1)$, $\ldots$ $\sum_{k=0}^{\lceil \frac{n-1}{6} \rceil}\xi(6k+5)$.

In this way using approximately $\lceil \log_{2} n \rceil$ steps all sequence $\{\xi(j)\}_{j=0}^{n-1}$ of complex envelopes of elementary pulses, forming a *SGBC*, is found.

IV. CONCLUSION

The above described method for synthesis of *SGBC* was realized as a computer program. With it the interval of code-lengths $14 \le n \le 25$ was examined. It was found the following *SGBS*'s:

$$\xi(14)=+1,-\omega^{2},-1,\omega^{2},\omega^{2},-1,\omega^{2},\omega,\omega^{2},-\omega^{2},-\omega,\omega^{2},\omega,+1; \quad (28)$$

$$\xi(15)=+1,-\omega^{2},\omega,\omega^{2},-\omega,\omega^{2},\omega^{2},\omega^{2},\omega,\omega^{2},$$
$$-\omega^{2},-\omega,\omega^{2},\omega,+1; \quad (29)$$

$$\xi(18)=+1,+1,\omega^{2},-\omega,\omega,\omega,+1,-\omega^{2},-\omega^{2},\omega,$$
$$\omega,-\omega,-1,+1,-\omega^{2},-\omega,\omega,+1. \quad (30)$$

The first two *SGBC* of length $n=14,\ 15$ respectively, are mentioned, but not shown in the literature [9]. The later one (see Eq. (30)) is unknown till now [3], [4], [5], [6], [7], [8], [9], [10].

At the end it ought to be mentioned that:

- the method for synthesis of *SGBC*, suggested in our paper, has good computational effectiveness;
- this method could be used successfully for finding of other signals, valuable for present communications.

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