Image Deblur in Case of Symmetric Kernel

Georgi Gluhchev[1], Mladen Savov[2], Vesselin Velichkov[3]

Abstract - **An algebraic approach to image restoration is suggested. It is assumed that the blur is caused by a Gaussian type kernel of size 3x3. It is shown that the matrix of the obtained system of linear equations can be presented as a symmetric block-matrix consisting of symmetric blocks of same structure.**

Keywords – **Image deblur, Gaussian kernel, Algebraic approach.**

I. Introduction

The restoration of blurred images is one of the most important problems in the image processing. Due to different factors images could be distorted to a significant degree. The restoration techniques depend on the character of the distortion and nature of its sources. The image blur for example may be caused by the relative movement between the camera and the object when capturing the image, or due to not properly adjusted optics.

In image processing the blur effects are thought as a convolution of an ideal image $a(x,y)$ with a kernel $k(x,y,\alpha,\beta)$ of specific parameters expressing the degradation process [1,2,3,4]. The regarded image $b(\alpha,\beta)$ is presented as the following convolution

$$b(\alpha,\beta)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}a(x,y)k(x,y,\alpha,\beta)dxdy\,.$$

In case of discrete images the above formula takes the form

$$b(i,j)=\sum_{m=1}^{n}\sum_{l=1}^{n}a(m,l)k(m,l,i,j) \qquad (1)$$

Using the properties of the Fourier transform (FT) of the convolution of two functions the deconvolution process is determined as an inverse FT of the ratio of the blurred image spectrum and kernel spectrum, i.e.

$$a(x,y)=F^{-1}(B(u,v)/K(u,v))\,,$$

where F^{-1} is the inverse Fourier transform.

However, this simple mathematical approach known as inverse filtering faces some difficulties in its practical realization due to small or zero values in the denominator at some frequencies.

If additive noise is present it could be amplified significantly. In any case the original image could not be exactly restored. In practice the kernel k(x,y,α,β) is usually of small size and its spectrum contains a lot of zeros. To avoid this, a non-zero value (constant or optimal with respect to some criterion) is added to the denominator. Thus a family of Fourier domain filters has been developed leading to different types of Winner filters, requiring either direct or recursive operations. For incoherent imaging systems least squares based restoration filters may produce negative image values. This is not the case with the maximum entropy based or Bayesian theorem based restoration filters [1].

In this paper a straightforward approach in case of restricted and symmetric Gaussian type kernel, based on the solution of an algebraic system of linear equations, is described. For this the presentation (1) is used, where the blur effect alongside the image borders is evaluated using only the neighbor pixels from the image. The special type of the kernel allows avoiding the evaluation of determinants and matrices of large size because the inverse matrix of the system is a block (but not circulant) matrix which could be analytically evaluated.

II. The Approach

Let the original, not blurred image, be of size nxn and the array of the pixels be denoted as $\boldsymbol{a}=\{a_{ij}\}$, (i,j = 1,2,…,$n$). Let the blur be caused by the kernel $\boldsymbol{k}$ of size 3x3 as shown in Fig. 1.

$$\boldsymbol{k}=\begin{Vmatrix}1&2&1\\2&4&2\\1&2&1\end{Vmatrix}$$

Fig. 1. Gaussian type kernel

We assume also, that the image is blurred placing the central element of $\boldsymbol{k}$ over every pixel of $\boldsymbol{a}$ and the blurred value is obtained as weighted sum of the pixels covered by the kernel. Thus, for a corner element only the element itself and its 3 neighbors will be used for the evaluation of the corresponding blurred value. For the border elements 6 pixels will be taken into account, while for the central elements 9 pixels will be used. The blurred image is of size nxn as well. Instead of using normalized weights we assume that the obtained blurred values are multiplied by 9, 12 and 16, respectively. As an illustration the following matrix of coefficients will be obtained for the case n = 3 (Fig.2):

[1]Georgi Gluhchev is with the Institute of Information Technologies, 2 Acad. G. Bonchev Str., Sofia, 1113, Bulgaria, E-mail: gluhchev@iinf.bas.bg
[2]Mladen Savov is a PhD student , E-mail: ihaiha99@abv.bg
[3]Vesselen Velichkov is a PhD student, E-mail: vesko@iinf.bas.bg

a_{11}	a_{12}	a_{13}	a_{21}	a_{22}	a_{23}	a_{31}	a_{32}	a_{33}
4	2	0	2	1	0	0	0	0
2	4	2	1	2	1	0	0	0
0	2	4	0	1	2	0	0	0
2	1	0	4	2	0	2	1	0
1	2	1	2	4	2	1	2	1
0	1	2	0	2	4	0	1	2
0	0	0	2	1	0	4	2	0
0	0	0	1	2	1	2	4	2
0	0	0	0	1	2	0	2	4

Fig. 2. Coefficient matrix of the system (1)

It could be presented as the following block-matrix **A**

$$\mathbf{A} = \begin{Vmatrix} 2\mathbf{B} & \mathbf{B} & \mathbf{0} \\ \mathbf{B} & 2\mathbf{B} & \mathbf{B} \\ \mathbf{0} & \mathbf{B} & 2\mathbf{B} \end{Vmatrix}$$

where **0** is a 3x3 zero matrix and **B** is the following 3x3 matrix

$$\mathbf{B} = \begin{Vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{Vmatrix}$$

For the determinant of **A** the following expression) holds

$$\det(\mathbf{A}) = 4^3 \det{}^3(\mathbf{B}).$$

For an arbitrary n the formula

$$\det(\mathbf{A}) = (n+1)^n \det{}^n(\mathbf{B}) \qquad (3)$$

will be obtained by induction.

In that case the matrix **B** looks as follows:

$$\mathbf{B} = \begin{Vmatrix} 2\ 1\ 0\ 0 \ldots 0\ 0\ 0 \\ 1\ 2\ 1\ 0 \ldots 0\ 0\ 0 \\ 0\ 1\ 2\ 1 \ldots 0\ 0\ 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \\ 0\ 0\ 0\ 0 \ldots 1\ 2\ 1 \\ 0\ 0\ 0\ 0 \ldots 0\ 1\ 2 \end{Vmatrix}$$

It is easily seen that det(**B**) = n+1. Thus

$$\det(\mathbf{A}) = (n+1)^{2n}. \qquad (4)$$

To solve the system (1) we have to evaluate the matrix ***c*** of adjuncts which is of size n^2 x n^2.

For $n = 2$ this matrix is as follows:

$$\boldsymbol{c} = \begin{Vmatrix} 36 & 18 & | & -18 & -9 \\ 18 & 36 & | & -9 & -18 \\ \hline -18 & -9 & | & 36 & 18 \\ -9 & -18 & | & 18 & 36 \end{Vmatrix}, \quad \text{or}$$

$$\mathbf{c} = \begin{Vmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{Vmatrix} = \begin{Vmatrix} -2C & C \\ C & -2C \end{Vmatrix} \text{ with } \quad C = \begin{Vmatrix} -18 & -9 \\ -9 & -18 \end{Vmatrix}$$

For $n = 3$ ***c*** is as follows:

$$\boldsymbol{c} = \begin{Vmatrix} 2304 & 1536 & 768 & | & 1536 & 1024 & 512 & | & 768 & 512 & 256 \\ 1536 & 3072 & 1536 & | & 1024 & 2048 & 1024 & | & 512 & 1024 & 512 \\ 768 & 1536 & 2304 & | & 512 & 1024 & 1536 & | & 256 & 512 & 768 \\ \hline 1536 & 1024 & 512 & | & 3072 & 2048 & 1024 & | & 1536 & 1024 & 512 \\ 1024 & 2048 & 1024 & | & 2048 & 4096 & 2048 & | & 1024 & 2048 & 1024 \\ 512 & 1024 & 1536 & | & 1024 & 2048 & 3072 & | & 512 & 1024 & 1536 \\ \hline 768 & 512 & 256 & | & 1536 & 1024 & 512 & | & 2304 & 1536 & 768 \\ 512 & 1024 & 512 & | & 1024 & 2048 & 1024 & | & 1536 & 3072 & 1536 \\ 256 & 512 & 768 & | & 512 & 1024 & 1536 & | & 768 & 1536 & 2304 \end{Vmatrix}$$

or

$$c = \begin{Vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{Vmatrix} = \begin{Vmatrix} 3C & 2C & C \\ 2C & 4C & 2C \\ C & 2C & 3C \end{Vmatrix}, \text{ with}$$

$$C = \begin{Vmatrix} 768 & 512 & 256 \\ 512 & 1024 & 512 \\ 256 & 512 & 768 \end{Vmatrix}.$$

Similar relations between the elements of ***c*** hold for n = 4, 5,… and they could be summarized as follows.

1. ***c*** is a symmetric matrix.
2. The elements of C_{11} are positive.
3. The sign ε_{ij} of C_{ij} changes according to the formula

$$\varepsilon_{ij} = (-1)^{n+1} sign(C_{i-1,j}) = (-1)^{n+1} sign(C_{i,j-1}), \qquad (5)$$

i.e., if $n = 2k + 1$ all sub-matrices C_{ij} consist of positive elements, otherwise the signs change alternatively.

4. The smallest absolute value in ***c*** is the value of $c_{n^2 1}$.
5. C is symmetric. Also, their elements are proportional to the smallest one and follow the rule:

$$C_{i,j} = 2C_{i+1,j} \text{ for } n - i < n - j + 1 \ (i,j = 1,2,\ldots,n) \qquad (6)$$

Thus, the elements of C, and therefore all the elements of ***c***, could be easily evaluated if the element $c_{n^2 1}$ is known. But it is easy to check that

$$C_{n^2 1} = (-1)^{n+1}(n+1)^{2(n-1)}. \qquad (7)$$

The obtained results show that the solution of the linear system (1) does not require $n^2 \, x \, n^2$ determinants of size $(n^2\text{-}1)$ x $(n^2\text{-}1)$ to be evaluated. Their values could be calculated straightforwardly according to the equations (3), (4), (5), (6) and (7).

The solution in a_{ij}, will be obtained according to the formula

$$a_{ij} = \frac{(-1)^{ni+j}}{\det(A)} \left(\sum_{m=1}^{n} (-1)^{nm} \sum_{p=1}^{n} (-1)^{p} C_{m(p),i(j)} b_{mp} \right) \quad (8)$$

Since the matrix $\boldsymbol{c}$ is of size $n^2 \ x \ n^2$ a lot of memory will be required to save it even in case of moderate value of n. However, the above formulated properties allow avoiding this problem. Knowing the smallest value $c_{n^2 1}$ all the elements $\boldsymbol{c}_{m(p),i(j)}$, where $m(p),i(j)$ stands for the element (p,j) of the block $C_{m,i}$ of $\boldsymbol{c}$, could be evaluated according to the formula

$$C_{m(p),i(j)} = \varepsilon_{mi} i(n-m+1) j(n-p+1) c_{n^2 1}, \quad (9)$$

where

$$\varepsilon_{mi} = \begin{cases} 1, & if \quad n = 2k+1 \\ (-1)^{m+i}, & otherwise \end{cases}.$$

Formula (8) is valid for $m \geq i$ and $p \geq j$. If $m < i$ or $p < j$, m and i or p and j respectively, have to exchange their places in (8).

Including the expression (9) in formula (8) and performing elementary calculations the following formula for a_{ij} will be obtained:

$$a_{ij} = ij \frac{(-1)^{ni+j}}{(n+1)^2} \left(\sum_{m=1}^{n} \varepsilon_{mi} (-1)^{nm} (n-m+1) \sum_{p=1}^{n} (-1)^{p} (n-p+1) b_{mp} \right) \quad (10)$$

where $i,j = 1,2,\ldots,n$.

III. Conclusion and Discussion

A straightforward algebraic approach to image deblur has been described provided an isotropic Gaussian type kernel of size 3x3 has caused the blur. The special kernel's type leads to direct evaluation of the determinant of the corresponding system of linear equations and its minors. Thus the heavy computational burden is overcome. This is the main advantage of the approach. The expression (3) shows that the system is always solvable and has a unique solution. The second advantage consists in the possibility of accurate restoration of the initially unblurred image if no round-off errors are present.

The third advantage stems from the integer arithmetic, which, in addition, will save computational resources and could allow for a fast parallel implementation.

However, the selected special type of the kernel is a shortcoming that will restrict the application of the approach.

It seems that similar results could be obtained with other isotropic kernels of larger size. This will be the next step in our investigation. Also, an interesting problem of great practical importance is the evaluation of the influence of the round-off errors and the possibility of its decrease. These errors are inevitable and stem from the integer output of the capturing devise. In [2] an illustration of the effect of round-off errors is present using one-dimensional signal of sinusoidal form of amplitude 25 presented with 30 points. The graph shows a significant offset from the original curve.

Acknowledgement

This investigation was supported by the Ministry of Education and Sciences in Bulgaria under contract No I-1302/2003.

References

[1] Jain, A. K., *Fundamentals of Digital Image Processing*, Prentice Hall, Englewood Cliffs, 1989

[2] Petrou M., P. Bosdogianni, *Image Processing*, John Wiley & Sons, Inc., 1999

[3] Parker J.R., *Algorithms for Image Processing and Computer Vision*, John Wiley & Sons, Inc., 1997

[4] Pratt, W. K. "*Digital Image Processing*" (Third Edition), John Wiley & Sons, Inc., 2001