Reflection Coefficient Equations for Tapered Transmission Lines

Zlata Cvetković[1], Slavoljub Aleksić, Bojana Nikolić

Abstract – **An approximation theory, based on the theory of small reflections, to predict the reflection coefficient response as a function of the impedance taper is applied in the paper. Obtained result is applied to a few common types of tapers. Transmission lines with exponential, triangular and Hermite taper are considered. All results are plotted using program package Mathematica 3.0. The presented results should be useful in solving reflection coefficient problems.**

Keywords – **Reflection coefficient, Voltage standing wave ratio (VSWR), Exponential, triangular and Hermite taper.**

I. INTRODUCTION

The analysis of nonuniform transmission lines has been a subject of interest of many authors. Uniform transmission lines can be used as impedance transformers depending on the frequency and length of the line [1]. The nonuniform lines have the advantage of wide-band impedance matching when they are used as impedance transformers and larger rejection bandwidths when they are used as filters [1, 2]. Some results for nonuniform exponential loss transmission line used as impedance transformer are presented in papers [3-5]. The paper [6] gives a solution in closed-form of the equation for value of arbitrary complex impedance transformed through a length of lossless, nonuniform transmission line with exponential, cosine-squared and parabolic taper.

In this paper we will derive an approximation theory, based on the theory of small reflections, to determine the reflection coefficient response as a function of the impedance taper. Obtained result is applied to transmission lines with exponential, triangular and Hermite taper [7]. For all examples the voltage standing wave ratio is plotted using program package Mathematica 3.0.

II. REFLECTION COEFFICIENT EQUATION FOR NONUNIFORM TRANSMISSION LINE

The traditional way of determining RF impedance was to measure voltage standing wave ratio (VSWR) using an RF detector, a length of slotted transmission line and a VSWR meter. VSWR is defined as the maximum value of the RF envelope over the minimum value of the RF envelope. As the probe detector was moved along the transmission line, the relative position and values of the peaks and valleys were noted on the meter. From these measurements, impedance could be derived. The procedure was repeated at different frequencies. Modern network analyzers measure the incident and reflected waves directly during a frequency sweep, and impedance results can be displayed in any number of formats.

Reflection loss is away to express the reflection coefficient in logarithmic terms (dB). The reflection coefficient is the ratio of the reflected signal voltage level to the incident signal voltage level. Reflection loss is the number of decibels that the reflected signal is below the incident signal. Reflection loss is always expressed as a positive number and varies between infinity for a load at the characteristic impedance and 0 dB for an open or short circuit.
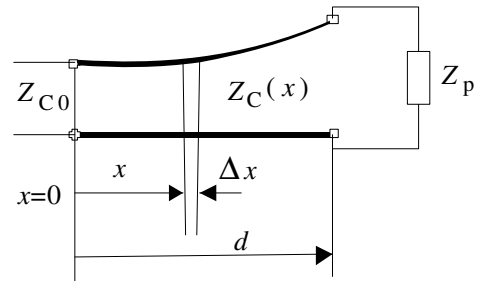

Fig. 1. A tapered transmission line matching section.

Let the transmission line shown in Fig. 1 be considered. The continuously tapered line can be modelled by large number of incremental sections of length Δx. One of these sections, connected at, has a characteristic impedance of $Z_C(x) + \Delta Z_C$ and one before has a characteristic impedance of $Z_C(x)$, as it is shown in Fig. 2. These impedance values are conveniently normalized by Z_{C0}. Then the incremental reflection coefficient from the step at distance x is given by

$$\Delta\Gamma = \frac{\left(Z_C(x) + \Delta Z_C\right) - Z_C(x)}{\left(Z_C(x) + \Delta Z_C\right) + Z_C(x)} \approx \frac{\Delta Z_C}{2 Z_C}. \qquad (1)$$

In the limit as $\Delta x \to 0$, it can be written as

$$d\Gamma = \frac{d Z_C}{2 Z_C} = \frac{1}{2}\frac{d(\ln Z_C(x))}{d x} d x \qquad (2)$$

since

$$\frac{d(\ln f(x))}{d x} = \frac{1}{f(x)}\frac{d(f(x))}{d x}.$$

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The corresponding incremental reflection coefficient at the input end can be written as follows

$$d\,\Gamma_{in} \approx e^{-j2\beta x}\,d\,\Gamma \ . \tag{3}$$

By using formula (2) of small reflections, the total reflection coefficient at the input end of the tapered section can be determined by summing all the partial reflections with their appropriate phase angles

$$\Gamma_{in} = \int_0^d d\,\Gamma_{in} = \frac{1}{2}\int_0^d e^{-j2\beta x}\frac{d}{d\,x}\left(\ln\frac{Z_C(x)}{Z_{C0}}\right)d\,x = |\Gamma_{in}|e^{-j\varphi}\ . \tag{4}$$

So if $Z_C(x)$ is known, reflection coefficient at $x = 0$ can be found as a function of frequency. On the other hand, if Γ_{in} is specified, then in principle $Z_C(x)$ can be determined.
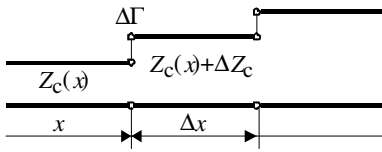

Fig. 2. Model for an incremental step change in impedance of the tapered line.

When a transmission line is terminated with an impedance, Z_P, that is not equal to the characteristic impedance of the transmission line, Z_C, not all of the incident power is absorbed by the termination. One part of the power is reflected back so that phase addition and subtraction of the incident and reflected waves creates a voltage standing wave pattern on the transmission line. The ratio of the maximum to minimum voltage is known as the voltage standing wave ratio (VSWR) and successive maxima and minima are spaced by 180°.

If the equation for reflection coefficient is solved for the VSWR, it is found that

$$\text{VSWR} = \frac{1 + |\Gamma_{in}|}{1 - |\Gamma_{in}|}\ . \tag{5}$$

The reflection or return loss is related through the following equation

$$\text{Reflection loss} = 20\ \log_{10}|\Gamma_{in}|\ \text{dB}\ . \tag{6}$$

Reflection loss is a measure in dB of the ratio of power in the incident wave to that in the reflected wave, and as defined above always has a positive value.

Also of considerable interest is the mismatch or insertion loss. This is a measure of how much the transmitted power is attenuated due to reflection. It is given by the following equation:

$$\text{Insertion loss} = -10\ \log_{10}\left(1 - |\Gamma_{in}|^2\right)\text{dB}\ . \tag{7}$$

Transmission line attenuation improves the VSWR of a load or antenna. Therefore, if you are interested in determining the performance of antennas, the VSWR should always be measured at the antenna connector itself rather than at the output of the transmitter. Transmission lines should have their insertion loss (attenuation) measured in lieu of VSWR, but VSWR measurements of transmission lines are still important because connection problems usually show up as VSWR spikes.

III. TAPERED TRANSMISSION LINES

In this paper, three types of nonuniform transmission lines with exponential, triangular and Hermite function taper are considered.

A. Exponential taper

Along an exponential taper, the impedance is changing exponentially with distance,

$$Z_C(x) = Z_{C0}\,e^{k\,x}, \text{ for } 0 < x < d \tag{8}$$

as indicated in Fig. 3. At $x = 0$ we have $Z_C(0) = Z_{C0}$. At $x = d$ we wish to have $Z_C(d) = Z_{Cd} = Z_{C0}e^{kd}$, what determines the constant

$$k = \frac{1}{d}\ln M \ , \tag{9}$$

where k is a taper coefficient and

$$M = \frac{Z_{Cd}}{Z_{C0}} \tag{10}$$

is a taper ratio. Z_{C0} and Z_{Cd} are the characteristic impedances of the transmission line at the left (source) and right (load) sides, respectively.

From (4), the total reflection coefficient at the input end is found as

$$\Gamma_{in} = \frac{1}{2}\int_0^d e^{-j2\beta x}\frac{d}{d\,x}\left(\ln e^{k\,x}\right)d\,x =$$

$$= \frac{1}{2}\ln\frac{Z_{Cd}}{Z_{C0}}\,e^{-j\beta d}\,\frac{\sin\beta d}{\beta d} = |\Gamma_{in}|e^{-j\varphi}\ . \tag{11}$$

B. Triangular taper

Characteristic impedance along triangular taper changes as

$$Z_C(x) = Z_{C0}\,e^{2\left(\frac{x}{d}\right)^2\ln\frac{Z_{Cd}}{Z_{C0}}}, \text{ for } 0 < x < d/2 \tag{12}$$

$$Z_C(x) = Z_{C0}\,e^{\left[4\frac{x}{d} - 2\left(\frac{x}{d}\right)^2 - 1\right]\ln\frac{Z_{Cd}}{Z_{C0}}}, \text{ for } d/2 < x < d\ ,$$

as it is presented in Fig. 3.

Evaluating reflection coefficient from (4) gives

$$\Gamma_{in} = \frac{1}{2}\ln M \ e^{-\mathrm{j}\beta d}\left[\frac{\sin \beta d}{\beta d}\right]^2 = |\Gamma_{in}|e^{-\mathrm{j}\varphi} \qquad (13)$$

C. Hermite taper

In the case of Hermite taper, impedance of the transmission line varies with distance x as follow

$$Z_C(x) = Z_{C0}\ e^{(k\,x)^2}, \text{ for } 0 < x < d \qquad (14)$$

The resulting reflection coefficient response is given by (4)

$$\Gamma_{in} = \ln M \ \frac{e^{-\mathrm{j}2\beta d}\left(1 + \mathrm{j}2\beta d - e^{\mathrm{j}2\beta d}\right)}{(2\beta d)^2} = |\Gamma_{in}|e^{-\mathrm{j}\varphi}. \qquad (15)$$

Fig. 3 shows the impedance variations for the exponential, triangular and Hermite tapers of the same taper ratio, $M = 3$.
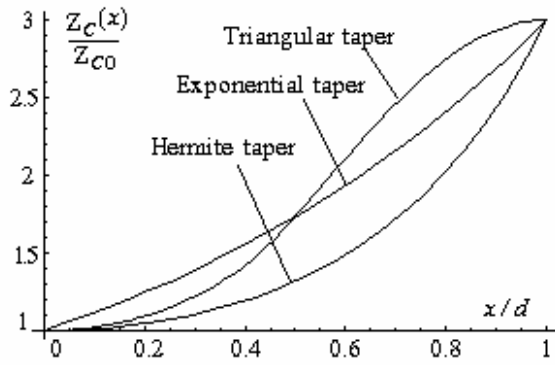
Fig. 3. Impedance variations for the exponential, triangular and Hermite tapers of the same taper ratio, $M = 3$.

IV. NUMERICAL RESULTS

According to the analysis presented above for exponential, triangular and Hermite transmission line, different calculations are done.
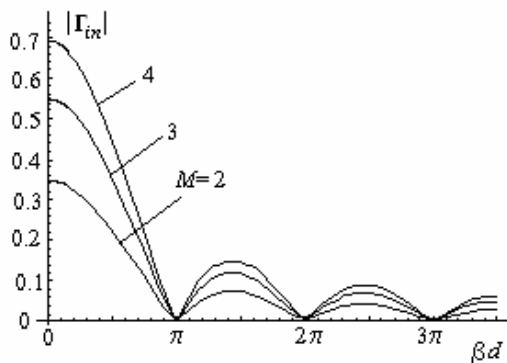
Fig. 4. Resulting reflection coefficient magnitude response on exponential line for different taper ratio.
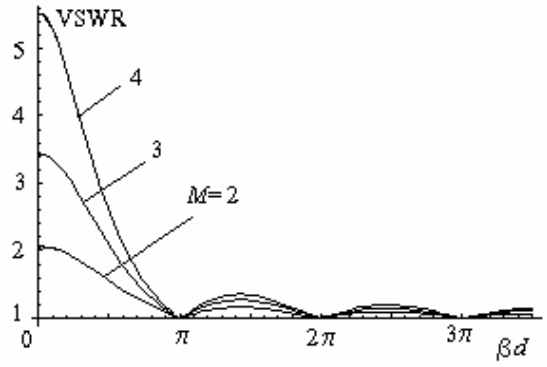
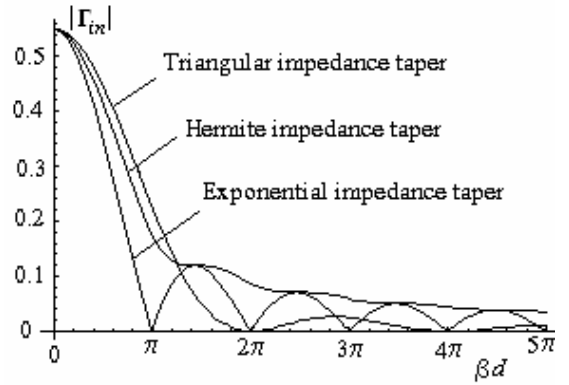Fig. 5. Resulting VSWR magnitude response on exponential line for different taper ratio.

Fig. 6. Resulting reflection coefficient magnitude versus frequency for the exponential, triangular and Hermite taper.
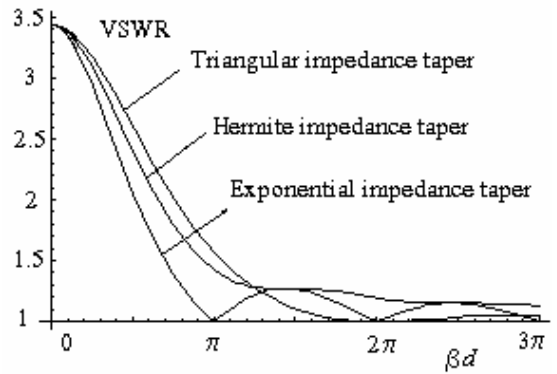
Fig. 7. Resulting reflection coefficient magnitude versus frequency for the exponential, triangular and Hermite taper.

Fig. 4 shows resulting reflection coefficient magnitude response on exponential line for different taper ratio $M = 2, 3, 4$. For the same taper ratios on exponential line, Fig. 5 presents VSWR magnitude response.

In order to compare obtained results, we consider an exponential, triangular and Hermite taper transmission line, which are used at the same taper ratio $M = 3$, as impedance transformers from $Z_{C0} = 100\,\Omega$ to $Z_{Cd} = 300\,\Omega$.

Resulting reflection coefficient magnitude versus frequency for the exponential, triangular and Hermite transmission line taper is shown in Fig. 6.

In Fig. 7, the voltage standing wave ratio (VSWR) of these nonuniform transmission lines is shown.

All the figures are plotted using program package Mathematica 3.0.

V. CONCLUSION

The modern network analyzer system sweeps very large frequency bandwidths and measures the incident power, P_i, and the reflected power, P_r. Because of the considerable computing power in the network analyzer, the reflection loss is calculated from the equation given previously, and displayed in real time. Optionally, the VSWR can also be calculated from the reflection loss and displayed real time.

In this paper, by using small reflections, the total reflection coefficient at the input end of the tapered section is determined by summing all the partial reflections up these incremental reflections with their appropriate phase angles. For exponential transmission line, resulting reflection coefficient and VSWR magnitude response for different taper ratio are presented. In order to compare obtained results for exponential, triangular and Hermite transmission line,

resulting reflection coefficient and VSWR are shown for the same taper ratio $M = Z_{Cd} / Z_{C0} = 3$.

Obtained results are plotted using program package Mathematica 3.0.

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