2D Electrical Circuit Analysis by Gaussian Procedures

Miodrag V. Gmitrović and Biljana P. Stošić

Abstract – **In this paper two efficient procedures for solving *2D* electrical circuits based on Gaussian elimination procedure are proposed. In order to reconsider the efficiency of the proposed procedures they are compared with two known procedures and the Equivalent Thevenin Sources (*ETS*) method. The voltage vector and impedance matrix of the *ETS* can be calculated by the Gaussian backward elimination procedure.**

Keywords **– *2D* electrical circuit, Gaussian eliminations**

I. Introduction

Two-dimensional (*2D*) electrical circuit consisting of elements with lumped parameters can be used for modeling of different physical processes. Some of these processes are propagation in microwave transmission lines [1-5], propagation in connections in microelectronic circuits (*RC* transmission lines) [6-7], superconductivity problems [8], diffusion problems in nuclear reactors [9] and etc. A complex *2D* electrical circuit can be modeled in different ways and solved in either time or frequency domain. The main problem is solving the system of linear equations $\mathbf{YU} = \mathbf{I}$, where $\mathbf{I}$ is a source vector, $\mathbf{U}$ is a vector of unknown voltages and $\mathbf{Y}$ is an admittance matrix of the *2D* electrical circuit. It is a very important to solve that equation system by using as less as possible arithmetic operations. There are some different procedures for solving such kind of systems and one of the most used is standard Gaussian elimination procedure (*GSE*) [9-11].

In the previously published papers [1,2], the Equivalent Thevenin Source (*ETS*) method is given. The analysis of *2D* circuit is based on decomposition of the complex circuit structure into cascade-connected ladder subnetworks with *2L* ports. For each ladder subnetwork corresponding transmission matrices are counted and all previously subnetworks are then substituted by its *L ETS*. They represent now the excitation of the next ladder subnetwork. The input and output voltages can be found by successive application of this procedure.

A *2D* electrical circuit can be represented as cascade connection of $2L$ port networks by diakoptics procedure [9], as shown in this paper. Admittance matrix $\mathbf{Y}$ is formed by appropriate choice of unknown voltages. The matrix is a tridiagonal matrix with non-zero submatrices on the main diagonal and on the first diagonals above and below the main diagonal and zero submatrices elsewhere. In order to solve the equation system with this tridiagonal matrix, two efficient procedures based on *GSE* procedure are proposed. The first procedure is Gaussian backward elimination (*GBE*) and the second one is Gaussian direct elimination (*GDE*). The voltage vector $\mathbf{U}_T$ and the impedance matrix $\mathbf{Z}_T$ of *ETS* for whole *2D* electrical circuit can be obtained by *GBE* procedure. In order to reconsider the efficiency of the suggested procedures (*GBE* and *GDE*) they are compared with *GSE* procedure, *ETS* method and the procedure that involves direct inversion of admittance matrix $\mathbf{Y}$ (*INV*).

Miodrag V. Gmitrović and Biljana P. Stošić are with the Faculty of Electronic Engineering, University of Niš, Aleksandra Medvedeva 14, 18000 Niš, Serbia and Montenegro.

E-mails: gmitrovic@elfak.ni.ac.yu and bilja@elfak.ni.ac.yu

II. Gaussian Procedures

In Fig.1 a *2D* electrical circuit with losses is shown. The circuit can be excited by several real voltage sources and terminated by several real loads.

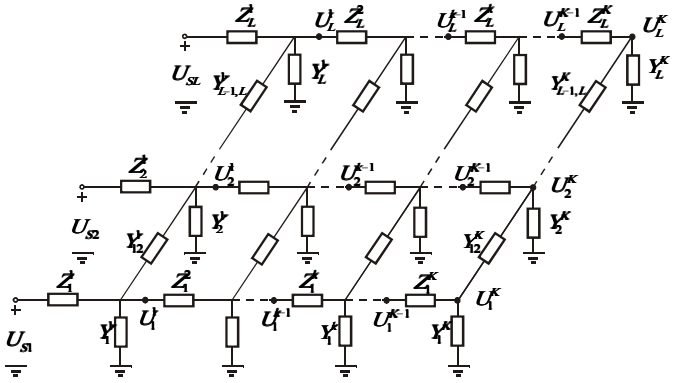

Fig. 1. A *2D* electrical circuit.

A *2D* electrical circuit with known voltage sources at the input ports and all immittances is observed. The goal is to count the node voltages at the output ports.

If the current sources at the input ports are known $I_{Sl} = U_{Sl} / Z_l^1$ than the unknown node voltages U_l^k, $k = 1,2,...,K$, $l = 1,2,...,L$, can be counted from the matrix equation system

$$\mathbf{Y}_K \cdot \mathbf{U}_K = \mathbf{I}_K . \tag{1}$$

In the previous equation system, the current vector is

$$\mathbf{I}_K = \left[\mathbf{I}^1 \quad \mathbf{I}^2 \quad \dots \quad \mathbf{I}^K\right]^T = \left[\mathbf{I}_S \quad \mathbf{0} \quad \dots \quad \mathbf{0}\right]^T , \tag{2}$$

where

$$\mathbf{I}^1 = \mathbf{I}_S = \left[I_{S1} \quad I_{S2} \quad \dots \quad I_{SL}\right]^T \tag{3}$$

and $\mathbf{0}$ is a zero vector.

The voltage vector is

$$\mathbf{U}_K = \left[\mathbf{U}^1 \quad \mathbf{U}^2 \quad \dots \quad \mathbf{U}^K\right]^T \tag{4}$$

and the admittance matrix is

$$\mathbf{Y}_K = \begin{bmatrix} \mathbf{Y}_{11}^1 & \mathbf{Y}_{12}^1 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{Y}_{21}^2 & \mathbf{Y}_{22}^2 & \mathbf{Y}_{23}^2 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y}_{32}^3 & \mathbf{Y}_{33}^3 & \mathbf{Y}_{34}^3 & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{Y}_{K-1,K-1}^{K-1} & \mathbf{Y}_{K-1,K}^{K-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{Y}_{K,K-1}^{K} & \mathbf{Y}_{K,K}^{K} \end{bmatrix} \quad (5)$$

where $\mathbf{0}$ is a zero matrix.

In Fig. 2, a l^{th} node of the k^{th} ladder subnetwork is depicted. The immittances Z_l^k, $l=1,2,...,L$, $Y_{l,l+1}^k$, $l=1,2,...,L-1$, correspond to the serial connections of resistors and inductors, and the admittances Y_l^k, $l=1,2,...,L$, $k=1,2,...,K$ correspond to the shunt connections of conductances and capacitors [1-3].

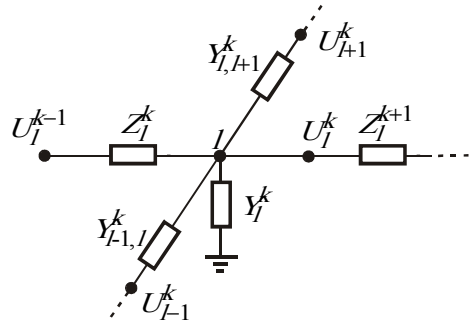

Fig. 2. The l^{th} node of the k^{th} ladder subnetwork.

The admittance matrix $\mathbf{Y}_K$ is a tridiagonal matrix. Also, their submatrices $\mathbf{Y}_{k,k}^k$, $k=1,2,...,K$, are tridiagonal matrices. The elements of these submatrices on the main diagonal are $1/Z_l^k + Y_l^k + 1/Z_l^{k+1} + Y_{l-1,l}^k + Y_{l,l+1}^k$, $l=1,2,...,L$ and $Y_{0,1}^k = Y_{L,L+1}^k = 0$. The elements on the first diagonal above the main diagonal are equal to those ones on the first diagonal below the main diagonal. Those elements are $-Y_{l,l+1}^k$, $l=1,2,...,L-1$. The other elements in these submatrices are zeros.

The submatrices of the matrix $\mathbf{Y}_K$ on the first diagonals above and below the main diagonal are equal $\mathbf{Y}_{k-1,k}^{k-1} = \mathbf{Y}_{k,k-1}^k$, $k=2,3,...,K$. These submatrices are diagonal matrices with the elements on the main diagonal $-1/Z_l^{k+1}$, $l=1,2,...,L$, $k=1,2,...,K-1$ and zeros elsewhere.

The system of matrix equations (1) can be solved by *GSE* [8-10] and in that case the admittance matrix $\mathbf{Y}_K$ is treated as full matrix. The matrix $\mathbf{Y}_K$ (5) has a lot of zero submatrices, and because of that standard Gaussian procedure can be modified in order to use only non-zero submatrices. Two new algorithms that are more efficient for solving matrix equation system (1) are proposed in further text. The matrix equation system (1) can be observed as a set of K equation subsystems. Each subsystem has L linear equations.

2.1. Gaussian backward elimination procedure (GBE)

The *GBE* procedure is a successive solving of the matrix equation system (1) starting from the last K^{th} linear subsystem, which corresponds to the voltage vector $\mathbf{U}^K$, and concluding with the first linear equation subsystem corresponds to the voltage vector $\mathbf{U}^1$. The electrical circuit from Fig.1, for taken voltage vectors $\mathbf{U}^k$, $k=1,2,\dots,K$, can be treated as cascade connection of $2L$ port subnetworks, Fig.3. A tridiagonal admittance matrix $\mathbf{Y}_K$ given by equation (5) can be reduced to a twodiagonal matrix by using *GBE* procedure [9-10]

$${}^K\mathbf{Y}_K = \begin{bmatrix} {}^1\mathbf{Y}_{11}^1 & \mathbf{Y}_{12}^1 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & {}^2\mathbf{Y}_{22}^2 & \mathbf{Y}_{23}^2 & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & {}^3\mathbf{Y}_{33}^3 & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & {}^{K-1}\mathbf{Y}_{K-1,K-1}^{K-1} & \mathbf{Y}_{K-1,K}^{K-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & {}^K\mathbf{Y}_{K,K}^K \end{bmatrix}. \quad (6)$$

The signs in the left upper corner indicate the iteration number. This is valid also for all relations in the further text.

The submatrices above the main diagonal are equal to those ones in the primary matrix (5). The submatrices on the main diagonal are counted by using next expressions

$${}^1\mathbf{Y}_{11}^1 = \mathbf{Y}_{11}^1, \quad (7)$$

$$\mathbf{M}_{k,k-1} = \mathbf{Y}_{k,k-1}^k \cdot \left({}^{k-1}\mathbf{Y}_{k-1,k-1}^{k-1}\right)^{-1}, \quad (8)$$

$${}^k\mathbf{Y}_{k,k}^k = \mathbf{Y}_{k,k}^k - \mathbf{M}_{k,k-1} \cdot \mathbf{Y}_{k-1,k}^{k-1}, \quad (9)$$

where $k=2,3,...,K$.

The current vectors can be counted by the relations

$${}^1\mathbf{I}^1 = \mathbf{I}^1, \quad (10)$$

$${}^k\mathbf{I}^k = {}^{k-1}\mathbf{I}^k - \mathbf{M}_{k,k-1} \cdot {}^{k-1}\mathbf{I}^{k-1}, \quad (11)$$

where $k=2,3,...,K$. Having on mind, the current vector $\boldsymbol{I}_K$ given by relation (2), it can be concluded that the current vector ${}^{k-1}\mathbf{I}^k \equiv \mathbf{0}$ for $k=2,3,...,K$. So, the relation (11) can be written as

$${}^k\mathbf{I}^k = -\mathbf{M}_{k,k-1} \cdot {}^{k-1}\mathbf{I}^{k-1}. \quad (12)$$

Starting by output ports of the *2D* electrical circuit, the voltage vectors can be counted by the next relations

$$\mathbf{U}^K = \left({}^K\mathbf{Y}_{K,K}^K\right)^{-1} \cdot {}^K\mathbf{I}^K, \quad (13)$$

$$\mathbf{U}^{k-1} = \left({}^{k-1}\mathbf{Y}^{k-1}_{k-1,k-1}\right)^{-1} \cdot \left({}^{k-1}\mathbf{I}^{k-1} - \mathbf{Y}^{k-1}_{k-1,k} \cdot \mathbf{U}^{k}\right), \quad (14)$$

where $k = K, K-1,...,3,2$.

A *2D* electrical circuit, Fig. 1, can be represented as cascade connection of $2L$ port subnetworks as shown in Fig. 3.

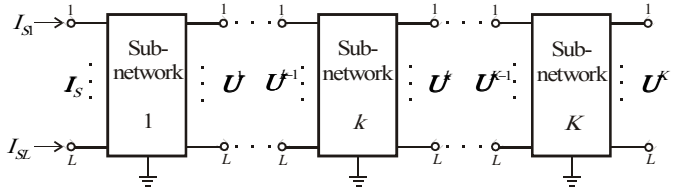

Fig. 3. Cascade connection of subnetworks with *2L* ports.

Cascade connection of *K* ladder subnetworks shown in Fig. 3, can be presented by *ETS* [2] as shown in Fig. 4.

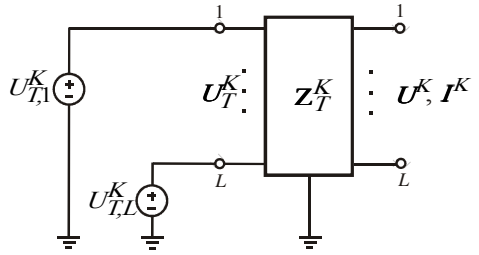

Fig. 4. *ETS* of voltage $\mathbf{U}^K_T$ and impedance $\mathbf{Z}^K_T$.

From the Fig.4 can be concluded that the voltage vector at the output ports is

$$\mathbf{U}^K = \mathbf{U}^K_T - \mathbf{Z}^K_T \cdot \mathbf{I}^K, \quad (15)$$

where $\mathbf{U}^K_T$ is the voltage vector and $\mathbf{Z}^K_T$ is the impedance matrix of the *ETS*. The impedance matrix of the *ETS*

$$\mathbf{Z}^K_T = ({}^K\boldsymbol{Y}^K_{K,K})^{-1} \quad (16)$$

is counted using the submatrix ${}^K\boldsymbol{Y}^K_{K,K}$ from the matrix ${}^K\mathbf{Y}_K$ (6). The voltage vector $\mathbf{U}^K_T$ for $\mathbf{I}^K = \mathbf{0}$ is equal to the voltage vector $\mathbf{U}^K$ obtained by relation (13). The calculated voltage vector $\mathbf{U}^K_T$ and impedance matrix $\mathbf{Z}^K_T$ can be used as inputs for the next ladder subnetwork in cascade connection. In this way can be solved cascade connection of ladder subnetworks with different number of ports. Such type of subnetworks can be used for successful analysis of different discontinuities in microwave transmission lines.

2.2. Gaussian direct elimination procedure (GDE)

The *GDE* procedure is a successive solving of the matrix equation system (1) starting from the first linear equation subsystem corresponds to the voltage vector $\mathbf{U}^1$ and concluding with the last K^{th} linear subsystem which corresponds to the voltage vector $\mathbf{U}^K$. In other words, *GDE* procedure is used for calculating the voltages from the matrix system (1) starting from the voltage vector $\mathbf{U}^1$. That vector presents the voltages on the output ports of the first subnetwork in cascade connection, Fig. 3. By the *GDE* procedure, the matrix $\mathbf{Y}_K$ is reduced to twodiagonal matrix of shape

$${}^K\mathbf{Y}_K = \begin{bmatrix} {}^K\mathbf{Y}^1_{11} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{Y}^2_{21} & {}^{K-1}\mathbf{Y}^2_{22} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y}^3_{32} & {}^{K-2}\mathbf{Y}^3_{33} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & {}^2\mathbf{Y}^{K-1}_{K-1,K-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{Y}^K_{K,K-1} & {}^1\mathbf{Y}^K_{K,K} \end{bmatrix} \quad (17)$$

The matrices below the main diagonal are equal to those ones in the primary matrix (5). The matrices on the main diagonal are counted by next expressions

$${}^1\mathbf{Y}^K_{K,K} = \mathbf{Y}^K_{K,K}, \quad (18)$$

$$\mathbf{M}_{k,k+1} = \mathbf{Y}^k_{k,k+1} \cdot \left({}^{K-k}\mathbf{Y}^{k+1}_{k+1,k+1}\right)^{-1}, \quad (19)$$

$${}^{K-k+1}\mathbf{Y}^k_{k,k} = \mathbf{Y}^k_{k,k} - \mathbf{M}_{k,k+1} \cdot \mathbf{Y}^{k+1}_{k+1,k}, \quad (20)$$

where $k = K-1, K-2,...,2,1$.

The voltages are here counted starting from the voltage vector on input ports by using the next relations

$$\mathbf{U}^1 = \left({}^K\mathbf{Y}^1_{11}\right)^{-1} \cdot \mathbf{I}^1, \quad (21)$$

$$\mathbf{U}^k = -\left({}^{K-k+1}\mathbf{Y}^k_{k,k}\right)^{-1} \cdot \mathbf{Y}^k_{k,k-1} \cdot \mathbf{U}^{k-1}, \quad (22)$$

where $k = 2,3,..., K$.

The *GDE* procedure in regards of the *GBE* procedure does not perform changes in current vector and the matrix equation system (1) is solved with less number of arithmetic operations.

III. EFFICIENCY ANALYSIS

In order to reconsider the efficiency of the suggested procedures (*GBE* and *GDE*), *2D* electrical circuits of various complexities are solved. The equivalent circuit composed as cascade-connected multi-port subnetworks shown in Fig.3 is analysed. The number of input ports is $L = 10$ and the number of cascade-connected subnetworks is $K = 1,2,...,200$. In this case, the largest equation system has 2000 unknown voltages. The efficiency-testing program is done in MATLAB [12] on PC 2.4 *GHz*.

Graphs of time needed for solving the equation system versus the number of networks in cascade connection K, are depicted in Fig.5. The time needed for the forming of admittance matrix $\boldsymbol{Y}_K$ of the *2D* circuit is included in the time needed for solving the equation system for all procedures. The MATLAB built-in functions are used for standard Gaussian

elimination procedure (*GSE*), i.e. $\mathbf{U} = \mathbf{Y} \setminus \mathbf{I}$, and matrix inversion procedure (*INV*), i.e. $\mathbf{U} = \text{inv}(\mathbf{Y}) \cdot \mathbf{I}$. The expressions given in the papers [1-3] are used for the *ETS* method (*ETS*).

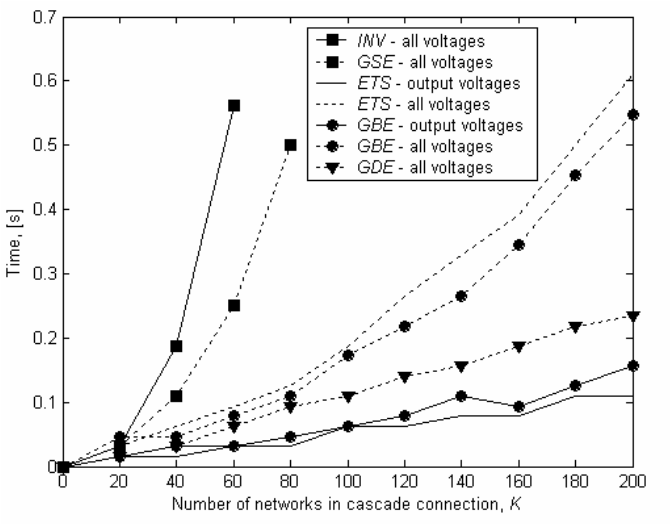

Fig. 5. Curve efficiency graph.

It can be inferred that the matrix inversion procedure requires the longest time needed for solving the equation system. In that case, a full admittance matrix $\mathbf{Y}_K$ is solved. The full admittance matrix is also solved by Gaussian standard procedure, but it is more efficient than the matrix inversion procedure. In MATLAB, the computations involving the backslash operator (\ ) require less computer time, less memory and have better error detection properties than the other one which forms the direct inverse (inv).

In the case when voltages in all nodes $\mathbf{U}_K$ are to be calculated, the most efficient method of *INV*, *GSE*, *ETS*, *GBE* and *GDE*, is the suggested procedure *GDE* since it doesn't require the calculation of the current vector $\mathbf{I}_K$. In the case when only output voltages $\mathbf{U}^K$ are to be calculated, the suggested procedure *GBE* and the *ETS* method require almost the same time for solving *2D* circuit. The *ETS* method doesn't require the calculation of the current vectors.

It is important to say that *GDE* procedure requires calculation of all node voltages in order to get the voltage vector $\mathbf{U}^K$ at the output ports. The essential advantage of the suggested *GBE* procedure is direct calculation of the output voltage vector without calculation of voltages in all circuit nodes. The other advantage of the *GBE* procedure is direct calculation of the *ETS* elements, i.e. the voltage vector $\mathbf{U}_T^K$ and the impedance matrix $\mathbf{Z}_T^K$.

IV. CONCLUSION

Two efficient procedures for solving matrix equation systems (1), which admittance matrix is a tridiagonal matrix, are proposed in this paper. Such admittance matrix is found for the *2D* electrical circuits, Fig. 1, represented as cascade-connected ladder $2L$ port subnetworks with voltages assigned as shown in Fig. 3.

It is shown that in the case when all voltages are to be calculated, the most efficient procedure of all is the *GDE* procedure. The most common case in practice is the calculation of the output voltages $\mathbf{U}^K$ only. In that case, the most efficient procedures are the *GBE* procedure and the *ETS* method [2]. The *GBE* procedure can be used for calculating the voltage vector $\mathbf{U}_T^K$ and the impedance matrix $\mathbf{Z}_T^K$ of the known *2D* electrical circuit. The suggested *GBE* procedure can be used successfully for solving complex microwave circuits containing transmission lines.

REFERENCES

[1] B. P. Stojanović and M. V. Gmitrović, "Modelling and Analysis of Cascade-Connected Planar Transmission Lines", *XII International Symposium on Theoretical Electrical Engineering - ISTET'03*, Poland, Warsaw, July 6-9, 2003, Volume II, pp. 473-476.

[2] B. P. Sto{i}, *Frequency Analysis of Planar Microwave Circuits by Thevenin Source Method,* Master thesis, Faculty of Electronic Engineering, University of Ni{, Ni{, February 2004, (in Serbian).

[3] B. P. Stošić and M. V. Gmitrović, "Frequency Analysis of Cascade-Connected Planar Transmission Lines by *ETS* Method", *Microwave Review*, No. 1, Vol. 10, June 2004, Serbia and Montenegro, pp. 20-29.

[4] W. K. Gwarek, "Analysis of Arbitrarily Shaped Two-Dimensional Microwave Circuits by Finite-Difference Time-Domain Method", *IEEE Transactions on Microwave Theory and Techniques,* Vol.36, No.4, April, 1988, pp. 738-744.

[5] L. N. Merugu and V. F. Fusco, "Concurent Network Diakoptics for Electromagnetic Field Problems", *IEEE Transactions on Microwave Theory and Techniques,* Vol.45, No.7, July, 1997, pp. 1086-1092.

[6] W. - K. Chen, *The Circuits and Filters Handbook*, CRC Press, 1995.

[7] M. V. Gmitrović, "Efikasan postupak za izračunavanje odziva u *RC* mrežama oblika stabla", *XXXI Jugoslovenska konferencija ETAN*, str. III.283-III.289, Bled, 1987 (in Serbian).

[8] C. E. Tong, L. Chen and R. Blundell, "Theory of Distributed Mixing and Amplification in a Superconducting Quasi-Particle Nonlinear Transmission Line", *IEEE Transactions on Microwave Theory and Techniques,* Vol.41, No.4, April, 1993, pp. 708-716.

[9] G. Kron, *Issledovanie sloænih sistem po åastàm – Diakoptika*, Nauka, Moskva, 1972.

[10] V. B. Litovski, *Projektovanje elektronskih kola*, DGIP "Nova Jugoslavija" - Vranje, Niš, 2000 (in Serbian).

[11] H. Kremer, *Numerical Analysis of Linear Networks and Systems*, Artech House, London, 1987.

[12] MATLAB – The Language of Technical Computing, Version 6.5.0.180913a, Release 13, June 18, 2002, The Math Works Inc. 1984-2002.