# Vector's model of spatial-temporal signal, interference and noise for simulating radar's optimal processor 

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#### Abstract

By construction mathematical model of spatial temporal signal as a vector, with same way the article will construct the model of spatial - temporal interference, noise and combinatorial interference-noise. In the result, one can define the spatial-temporal interference-noise covariance matrix. This is the key step for simulation optimal interference suppressive processor using in radar.


Keywords - Vector model of spatial-temporal signal; Interferen-ce-noise covariance matrix; Optimal interference suppressive processor.

## I. INTRODUCE

For increasing probability of target detection (such as predetective process), in modern radar systems one often apply the spatial - temporal optimal processor (STOP). The basic task of STOP is to combine the signals in such a way that the interference is reduced to the level of the thermal noise while the desired signal is preserved. Stated another way, the goal of STOP would like to maximize the output Signal to Interference plus Noise Ratio (SINR). The general schema of STOP is shown in Figure 1.1.


From this schema, we can see that the STOP consists of two basic steps: Optimal interference suppression and spatial temporal matched filter. Wherein, the optimal interference

[^0]suppression filter plays most importance role and is an essential step to increased SINR. By nature, it is a digital filter, knows as optimal interference suppressive filter and its frequency response has form of inverse interference spectrum. To satisfying the special requirement of this frequency response, the coefficients (or weights) of the digital filter are defined by some algorithm (often it is adaptive algorithm), that base on estimation of interference spectrum. By close mathematical provableness, one has gained result: The coefficients of the digital filter accurately equal components of inverse interference-noise covariance matrix $\mathbf{Q}^{-1}$. Therefore in simulative models of the STOP, one often use the block $\mathbf{Q}^{-1}$ instead of the optimal interference suppressive filter. For example, the simulative model of Auxiliary Channel Processor (ACP) shown on Figure 1.2 is a characteristic Spatial - Temporal Near-Optimal Processor. In this model, the optimal interference suppressive filter has been simplifying such as the first row of inverse of interference-noise covariance matrix.


For these reasons, the most important problem in simulative process of Spatial - Temporal Optimal Processor is definition of interference-noise covariance matrix $\mathbf{Q}$.

## II. Vector's Models of Spatial, Temporal and Spatial-TEMPORAL SignALS

The construction for mathematical models of signals is to organize and arrange data in some fixed order. The vector's model of signal consist of 2 parts: the scalar part is represent
for the length of this vector, which is frequently called module or amplitude and the directional part of vector is an unit vector (having unit norm), that is frequently called steered vector of signal.

In the spatial domain, by $[1,2]$ we have a vector of spatial signal, that is the set of received signals from series outputs of $N$ sensors, which is on a line with equal spacing between them (or uniform linear). It can be expressed as:

$$
\begin{equation*}
\mathbf{s}^{(K)}(n)=\sqrt{N} \cdot \mathbf{v}^{(K)}\left(f^{(K s)}\right) \cdot s(n) \tag{2.1}
\end{equation*}
$$

where: $s(n)$ is amplitude (scalar part) of the signal vector, received from any output of a arbitrary sensor at a time instant $n$; $f^{(K s)}$ is normalized spatial frequency of signal, which is defined by:

$$
\begin{equation*}
f^{(K s)}=\frac{d \cdot \sin \varphi_{s}}{\lambda_{0}} \tag{2.2}
\end{equation*}
$$

$d$ - uniform space between 2 successive sensors;
$\varphi_{s}$ - azimuth angle of arrival signal;
$\lambda_{0}$ - wavelength of carried signal;
The $\mathbf{v}^{(K)}\left(f^{(K s)}\right)$ is steering vector of spatial signal, that is a Vandermonde vector. In this case we can write:

$$
\mathbf{v}^{(K)}\left(f^{(K s)}\right)=\frac{1}{\sqrt{N}}\left[\begin{array}{llll}
1 & e^{-j 2 \pi f^{(K s)}} & \ldots & e^{-j 2 \pi(M-1) f^{(K s)}} \tag{2.3}
\end{array}\right]^{\prime}
$$

(Note: From (2.3) it can be deduce $\mathbf{v}^{(K)^{*}} \mathbf{v}^{(K)}=1$, or the norm of steering vector $\mathbf{v}^{(K)}$ equals unity);

Notation $K$ - refers to space;
Notation $s$ - refers to signal.
Likewise, in temporal domain, we can introduce the concepts about vector of temporal signal and steering vector of temporal signal. In accordance to [2], the reflective signal received from a moving target and defined at a time instant $n$ on output of any arbitrary sensor, supposed being on the first sensor, has the form:

$$
\begin{equation*}
s(n)=A \cdot e^{j 2 \pi n T_{p} f_{D_{s}}} \tag{2.4}
\end{equation*}
$$

where: $T_{P}$ is a pulse's period;
$f_{D s}$ is Doppler frequency of target's signal, which defined as:

$$
\begin{equation*}
f_{D}=\frac{2 v_{p}}{\lambda_{0}} \cos \varphi \tag{2.5}
\end{equation*}
$$

If set:

$$
\begin{equation*}
f^{(T s)}=T_{P} f_{D s}=\frac{f_{D s}}{f_{P}} \tag{2.6}
\end{equation*}
$$

Then $f^{\left(T_{s}\right)}$ is called the normalized temporal frequency of signal and now (2.4) can rewrite as:

$$
\begin{equation*}
s(n)=A . e^{j 2 \pi n f^{\left(T T_{3}\right)}} \tag{2.7}
\end{equation*}
$$

When this signal pass through a delay line consisted of $M-1$ tabs, what the lag of each tab equals $T_{p}$, it is correspond to phase rotation at an angle $-j 2 \pi T_{p} f_{D}$ (radian).

Thus, the input signal of delay line is:
$s_{1}^{(T)}(n)=s(n)=A \cdot e^{j 2 \pi n f^{\left(T_{3}\right)}}$. The signal after first lag is:
$s_{2}^{(T)}(n)=A \cdot e^{j 2 \pi(n-1) f^{\left(T_{T}\right)}}$. The signal after second lag is:
$s_{3}^{(T)}(n)=A . e^{j 2 \pi(n-2) f^{(T s)}}$. The signal after $M-1$-st lag is:
$s_{M}^{(T)}(n)=A . e^{j 2 \pi[n-(M-1)] f^{\left(T_{s}\right)}}$
Denoted $\mathbf{s}^{(T)}(n)$ to be the temporal signal vector, which comprised of output signals of all successive delay taps, and then we have:

$$
\begin{align*}
& \mathbf{s}^{(T)}(n)=\left[\begin{array}{llll}
s_{1}^{(T)}(n) & s_{2}^{(T)}(n) & \ldots & s_{M}^{(T)}(n)
\end{array}\right]^{\prime}= \\
& \quad=\left[\begin{array}{llll}
A \cdot e^{j 2 \pi n f^{(T s)}} & A \cdot e^{j 2 \pi(n-1) f^{(T s)}} & \ldots & A \cdot e^{j 2 \pi[n-(M-1)] f^{(T s)}}
\end{array}\right]^{\prime}= \\
& =A \cdot e^{j 2 \pi n f^{(T s)}}\left[\begin{array}{llll}
1 & e^{-j 2 \pi f^{(T s)}} & \ldots & e^{-j 2 \pi(M-1) f^{(T s)}}
\end{array}\right]^{\prime}= \\
& =\sqrt{M} \mathbf{v}^{(T)}\left(f^{(T s)}\right) \cdot s(n) \tag{2.8}
\end{align*}
$$

where: $\mathbf{v}^{(T)}\left(f^{\left(T_{s}\right)}\right)$ is temporal signal steering vector, in which:

$$
\mathbf{v}^{(T)}\left(f^{\left(T_{s}\right)}\right)=\frac{1}{\sqrt{M}}\left[\begin{array}{llll}
1 & e^{-j 2 \pi f^{\left(T_{s}\right)}} & \ldots & e^{-j 2 \pi(M-1) f^{\left(T_{s}\right)}} \tag{2.9}
\end{array}\right]^{\prime}
$$

(Note: From (2.9) it can be deduce $\mathbf{v}^{(T)^{*}} \mathbf{v}^{(T)}=1$, or the norm of $\mathbf{v}^{(T)}$ equals unity).

The term "steering" expresses a common meaning, which in detail as "rotating to a direction..." in the spatial domain or "reaching to frequency..." in the time domain.

By use of one dimension signal vectors above, we can construct the vector's model of spatial-temporal signal. From the steering vector of spatial signal (1.3) and steering vector of temporal signal (1.8) we have the steering vector of spatialtemporal signal corresponding to the normalized temporal frequency $f^{(K s)}$ and Doppler normalized frequency $f^{(T s)}$ :

$$
\begin{equation*}
\mathbf{v}^{(K T)}\left(f^{(K s)}, f^{(T s)}\right)=\mathbf{v}^{(K)}\left(f^{(K s)}\right) \otimes \mathbf{v}^{(T)}\left(f^{(T s)}\right) \tag{2.10}
\end{equation*}
$$

where $\otimes$ is Kronecker product of 2 vectors; notation ( $K T$ ) expresses the quantities in two-directional spatial-temporal domain. This vector is similar two correspond one-directional steering vector which has unit norm, that is $\mathbf{v}^{(K T)^{*}} \mathbf{v}^{(K T)}=1$. From (2.10) shown that spatial-temporal steering vector has dimension $N M \times 1$. With spatial-temporal steering vector $\mathbf{v}^{(K T)}$ we can construct the vector model of spatial-temporal signal, which has normalized temporal frequency $f^{(K s)}$ and normalized Doppler frequency $f^{(T s)}$ :

$$
\begin{equation*}
\mathbf{s}^{(K T)}(n)=\sqrt{N M} \cdot \mathbf{v}^{(K T)}\left(f^{(K s)}, f^{(T s)}\right) \cdot s(n) \tag{2.11}
\end{equation*}
$$

Note that, the time variance $n$ in (2.11) for quantities in two-dimension spatial-temporal domain "spreads" on an interval, which is defined:

- In space, it is the time for the signal passing over $N$ sensors: $T_{N}$
- In time, it is the time for the signal passing over ( $M-1$ ) delay taps: $(M-1) \tau$.

Normally, $(M-1) \tau \square T_{N}$, therefore the actual spreading time is $(M-1) \tau$. Due to interested quantities are examined at the same starting time, therefore the variance $n$ is not necessary. Thus, (2.11) can be rewritten:

$$
\begin{equation*}
\mathbf{s}^{(K T)}=\sqrt{N M} \cdot \mathbf{v}^{(K T)}\left(f^{(K s)}, f^{(T s)}\right) \cdot s \tag{2.12}
\end{equation*}
$$

The dimension of vector $\mathbf{s}^{(K T)}$ is $(N M \times 1)$.
The simulation algorithm of spatial-temporal signal vector followed (2.12) need input data, which comprise amount of sensors $N$, delay taps $M$, signal amplitude $s$, normalized spatial frequency $f^{(K s)}$ and normalized Doppler frequency $f^{(T s)}$. Where, the steering vector is defined follow (2.10).

## III. VECTOR'S MODEL OF SPATIAL-TEMPORAL INTERFERENCE AND NOISE. CONSTRUCTION COVARIANCE MATRIX OF INTERFERENCE - NOISE

Suppose that there are $C$ interference sources, divided into:

- $C_{K}$ spatial interference sources, corresponding to the spatial frequencies $f_{p}^{(K i)}$ with ( $p=1 \div C_{K}$ );
- $C_{T}$ temporal interference sources, corresponding to the temporal frequencies $f_{q}^{(T i)}$ with $\left(q=1 \div C_{T}\right)$;
where, notation $i$ expresses the quantities related to interference and $C=C_{K}+C_{T}$.

With interference, the spatial and temporal frequencies have relation following (2.6) ( $\varphi$ corresponding to the spatial frequency defined in (2.2), $f_{D}$ corresponding to temporal frequency in (2.6)). Therefore, each spatial frequency $f_{p}^{\left(K_{i}\right)}$ corresponds to the temporal frequency $f_{p}^{(T i)}$ and each temporal frequency $f_{q}^{(T i)}$ corresponds to spatial frequency $f_{q}^{\left(K_{i}\right)}$. In the other words, $C$ interference sources correspond to $C$ couple of spatial frequency $f_{l}^{(K i)}$ and temporal frequency $f_{l}^{(T i)}$, with $l=1 \div C$.

It can be deduce that, similarly to the vector of spatialtemporal signal (2.12), we can construct the vector of spatial temporal interference for an interference source $l$, amplitude $\sigma_{i l}$. It is written as:

$$
\begin{equation*}
\mathbf{i}_{l}^{(K T)}=\sqrt{N M} \cdot \sigma_{i l} \cdot \mathbf{v}^{(K T)}\left(f_{l}^{(K i)}, f_{l}^{(T i)}\right) \tag{3.1}
\end{equation*}
$$

The dimension of vector $\mathbf{i}_{l}^{(K T)}$ is $(N M \times 1)$.
The general spatial - temporal interfe-rence vector of all interference sources is:

$$
\begin{equation*}
\mathbf{i}^{(K T)}=\sum_{l=1}^{C} \mathbf{i}_{l}^{(K T)} \tag{3.2}
\end{equation*}
$$

Assume that, the interference intensity on each of all directions (corresponding to all Doppler frequencies) is equal, that is $\sigma_{i l}=\sigma_{i}$ with all $l$. Then (3.2) is now expanded:

$$
\begin{equation*}
\mathbf{i}^{(K T)}=\sqrt{N M} \cdot \sigma_{i} \sum_{l=1}^{C} \mathbf{v}^{(K T)}\left(f_{l}^{(K i)}, f_{l}^{(T i)}\right)=\sqrt{N M} \cdot \sigma_{i} \cdot v_{C}^{(K T)} \tag{3.3}
\end{equation*}
$$

where $\mathbf{v}_{C}^{(K T)}=\sum_{l=1}^{c} \mathbf{v}^{(K T)}\left(f_{l}^{(K i)}, f_{l}^{(T i)}\right)$ is general spatial - temporal interference steering vector with dimension $(N M \times 1)$. One element at $k^{\text {th }}$ row $(k=p, q)$ of this vector is detail expressed in form:

$$
\begin{equation*}
v_{C}^{(K T)}(p, q)=\sum_{l=1}^{C} e^{-j 2 \pi\left(p f_{l}^{(K i)}+q f_{l}^{(T)}\right)} \tag{3.4}
\end{equation*}
$$

where: $p=1 \div N$ and $q=1 \div M$.
The vector of spatial - temporal noise is the set of energy existing over all spatial - temporal channels, is also constructed in way based on (2.12). Assumed that the white noise having amplitude $\sigma_{n}$ contributes equally on all spatial - temporal channels, and then the vector of spatial - temporal noise has the form:

$$
\begin{equation*}
\mathbf{n}^{(K T)}=\sqrt{N M} \cdot \sigma_{n} \cdot \mathbf{v}^{(K T)}\left(f^{(K n)}, f^{(T n)}\right) \tag{3.5}
\end{equation*}
$$

with dimension $(N M \times 1)$.
The simulation algorithm of the general spatial - temporal interference vector carrying out by (3.3) requires input data, which composed of amount sensors $N$, delay taps $M$, isotropic interference amplitude $\sigma_{i}$ and $C$ normal spatial frequency of the interference $f_{l}^{(K i)}$. The normalized Doppler frequencies of the interference $f_{l}^{(T i)}$ are defined through normalized spatial frequency of the interference $f_{l}^{(K i)}$ by (2.2), (2.5) and (2.6). In which, steering vector is still defined by (2.10)

The simulation algorithm of the vector of spatial - temporal noise carrying out by (3.5) requires input data, which composed of amount sensors $N$, delay taps $M$, isotropic noise amplitude $\sigma_{n}$, normalized spatial frequency $f^{(K n)}$ and normalized Doppler frequency $f^{(T n)}$. In which, steering vector is still defined by (2.10).

The combinative spatial - temporal signal vector also commonly referred to as spatial - temporal data vector $\mathbf{x}^{(K T)}$ has the form:

$$
\begin{equation*}
\mathbf{x}^{(K T)}=\mathbf{s}^{(K T)}+\mathbf{i}^{(K T)}+\mathbf{n}^{(K T)} \tag{3.6}
\end{equation*}
$$

where: $\mathbf{s}^{(K T)}$ - the spatial - temporal signal vector;
$\mathbf{i}^{(K T)}$ - the general spatial- temporal interference vector
$\mathbf{n}^{(K T)}$ - the spatial - temporal noise vector.

Note that, all the spatial - temporal vectors shown in (3.6)
have the dimension corresponding to the signal vector that is equal to $N M \times 1$.

The summation of the interference and noise vector, denoted as $\mathbf{q}^{(K T)}$, is called the spatial - temporal interference noise vector:

$$
\begin{equation*}
\mathbf{q}^{(K T)}=\mathbf{i}^{(K T)}+\mathbf{n}^{(K T)} \tag{3.7}
\end{equation*}
$$

The spatial - temporal interference-noise covariance matrix $-\mathbf{Q}$, is defined as:

$$
\begin{equation*}
\mathbf{Q} \square E\left\{\mathbf{q}^{(K T)} \cdot\left(\mathbf{q}^{(K T)}\right)^{*}\right\} \tag{3.8}
\end{equation*}
$$

Thus $\mathbf{Q}$ has the dimension $N M \times N M$ and it is determined as followings.

Due to the linearity of the mathematical expectation operator and note that interference and noise are not correlative, therefore with respect to (3.8), we can expand as follow:

$$
\begin{align*}
& \mathbf{Q}=E\left\{\left(\mathbf{i}^{(K T)}+\mathbf{n}^{(K T)}\right)\left(\mathbf{i}^{(K T)}+\mathbf{n}^{(K T)}\right)^{*}\right\}= \\
& =E\left\{\mathbf{i}^{(K T)} \mathbf{i}^{(K T)^{*}}\right\}+E\left\{\mathbf{n}^{(K T)} \mathbf{n}^{(K T)^{*}}\right\}=\mathbf{Q}_{i}+\mathbf{Q}_{n} \tag{3.9}
\end{align*}
$$

where: $\mathbf{Q}_{i}$ - is the spatial - temporal interference covariance matrix;
$\mathbf{Q}_{n}$ - is the spatial - temporal noise covariance matrix.
*Determining the noise covariance matrix
$\mathbf{Q}_{n}=E\left\{\mathbf{n}^{(K T)} \mathbf{n}^{(K T)^{*}}\right\}:$
Replacing the spatial-temporal vector value in (3.5), we get:

$$
\begin{equation*}
\mathbf{Q}_{n}=\sigma_{n}^{2} \mathbf{I}=P_{n} \mathbf{I} \tag{3.10}
\end{equation*}
$$

where: $\quad \mathbf{I}$ - is unit diagonal matrix;

$$
P_{n} \text { - is the noise mean power. }
$$

Physically, we may clearly understand the meaning of (3.10), because of un-correlation of the noises in both spatial and temporal, therefore the only elements lying on diagonal are equal to 1 (due to the auto-correlation), the others are equal to 0 .

* Determining the interference covariance matrix
$\mathbf{Q}_{i}=E\left\{\mathbf{i}^{(K T)} \mathbf{i}^{(K T)^{*}}\right\}:$
Replacing the spatial - temporal interference vector value in (3.3), we get:

$$
\begin{equation*}
\mathbf{Q}_{i}=\sigma_{i}^{2} \cdot E\left\{\mathbf{v}_{C}^{(K T)} \cdot \mathbf{v}_{C}^{(K T)^{*}}\right\}=P_{i} \cdot E\left\{\mathbf{v}_{C}^{(K T)} \cdot \mathbf{v}_{C}^{(K T)^{*}}\right\} \tag{3.11}
\end{equation*}
$$

where: $P_{i}$ - is the interference mean power.
The solution to determine any one of elements of the interference covariance matrix is presented as following:

- As we knew, an element lying on the row $p \& q$ of the general steering vector has the form shown in (3.4).
- By similarly way, we may determine any one of elements
lying on the column $(g, h)$ of the vector $\mathbf{v}_{C}^{(K T)^{*}}$ :

$$
\begin{equation*}
v_{C}^{(K T)}(g, h)=\sum_{l=1}^{C} e^{j 2 \pi\left(g f_{l}^{(K i)}+h f_{l}^{(T)}\right)} \tag{3.12}
\end{equation*}
$$

with $g=1 \div N$ and $h=1 \div M$.

- From the (3.4) and (3.12), we can find out an element lying on row $(p, q)$ and at the column $(g, h)$ of the interference covariance matrix $\mathrm{Q}_{i}$ determined as form:

$$
\begin{align*}
Q_{i}(p, q ; g, h) & =P_{i} \cdot E\left\{v_{C}^{(K T)}(p, q) \cdot v_{C}^{(K T)^{*}}(g, h)\right\}= \\
& =P_{i} \cdot \sum_{l=1}^{C} e^{j 2 \pi\left((g-p) f_{l}^{(K i)}+(h-q) f_{l}^{(T i)}\right)} \tag{3.13}
\end{align*}
$$

## IV. Conclusion

In fact, mathematical models of spatial-temporal signals may be shown by alternative 3 methods:

The classically expression method that common used is analytic form. By this method, the mathematical model of signal is shown in form as a function. In the case of spatialtemporal signal, the analytic model is a multi-variable function (or multidimensional function). This method is very general and often uses to analyze any signal or system. But it has the basic weakness, that is: It can't be use in simulation, because can't construct the spatial-temporal interference-noise covariance matrix from it.

For multidimensional signals such as the spatial-temporal signals, we may apply matrix solution to mathematically simulate them. However, the interference-noise covariance matrix expressing the correlation among elements of the two matrices will have spatial dimension very large and become extremely complex. Therefore, this method is unprofitable to simulate spatial-temporal optimal processor.

Thus the expression of vector model provided in this article is rather simple to simulate process of interference suppression, using in spatial-temporal signal vector model. This affirms the special advantages of the spatial-temporal signal vector model.

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