Studies of Linear and Quadratic Approximations of GPS Satellite Radius Alteration in Time

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Abstract –**This paper presents some studies of linear and quadratic approximation of radius alteration of satellite (SV) in GPS using Taylor's series. Studies are made, using real data from satellites ephemeris message. The results for the best and the worst case are shown and the errors of approximations are obtained.**

Keywords – **GPS, approximation.**

I. INTRODUCTION

Presently, GPS is only fully operational satellite positioning system. The satellites broadcast ranging codes and navigation data. The navigation data provides the means for a receiver to determine the location of satellites, whereas the ranging codes enable the user's receiver to determine the propagation times of signals and thereby determine the satellite-to-user ranges.

The ephemeris message is a part of navigation data and contains the following orbital parameters [1, 2]:

- t_{0e} – Reference time of ephemeris;
- $\sqrt{a}$ – Square root of semi major axis;
- e – Eccentricity;
- i_0 – Inclination angle (at time t_{0e});
- Ω_0 – Longitude of the ascending node (at weekly epoch);
- ω – Argument of perigee (at time t_{0e});
- M_0 – Mean anomaly (at time t_{0e});
- di/dt – Rate of change of inclination angle;
- $\dot{\Omega}$ – Rate of change of longitude of the ascending node;
- Δn – Mean motion correction;
- C_{uc} – Amplitude of cosine correction to argument of latitude;
- C_{us} – Amplitude of sine correction to argument of latitude;
- C_{rc} – Amplitude of cosine correction to orbital radius;
- C_{rs} – Amplitude of sine correction to orbital radius;
- C_{ic} – Amplitude of cosine correction to inclination angle;
- C_{is} - Amplitude of sine correction to inclination angle;

These parameters are used to calculate the rest of satellite orbital parameters and coordinates: n-corrected mean motion, E-eccentric anomaly, ϕ-argument of latitude, r-corrected radius, (x, y, z)-satellite coordinates in Earth-Centered Earth-Fixed (ECEF) coordinate system, utilize a standard algorithm [1, 2, 3].

There is an indirect relationship between radius of satellite and time. Usually it is more convenient when this relation is direct [6]. An approximation is needed to convert indirect to direct dependence.

In this paper, studies of linear and quadratic approximation of radius for each satellite, are presented. A U-blox GPS receiver ANTARIS AEK-4P is used to collect the necessary ephemeris data.

II. APPROXIMATION OF RADIUS

In the standard algorithm the satellite radius r_{sk} is calculated by the following equation:

$$r_{sk} = a(1 - e\cos E_k) + C_{rs}\sin(2\phi_k) + C_{rc}\cos(2\phi_k)\,, \quad (1)$$

where index "k" denote the k^{th} satellite; E_k and ϕ_k are functions of time.

One of the most popular approximation techniques is representation of a function in a Taylor series [4, 5]. By expanding (1) in Taylor series about time instant $t=t_n$, for radius is obtained:

$$r_{sk}(t) \approx r_{sk}(t_n) + \frac{dr_{sk}}{dt}\Big|_{t=t_n}(t - t_n) + \frac{1}{2}\frac{d^2 r_{sk}}{dt^2}\Big|_{t=t_n}(t - t_n)^2 + \frac{1}{6}\frac{d^3 r_{sk}}{dt^3}\Big|_{t=t_n}(t - t_n)^3\,, \quad (2)$$

The close form for derivatives in (2) is found and they result in the following equation for satellite radius:

$$r_{ak}(t) \approx r_{sk}(t_n) + A(t - t_n) + B(t - t_n)^2 + C(t - t_n)^3\,, \quad (3)$$

where: A, B and C are approximation coefficients, calculated at the instant $t=t_n$ by the following equations:

$$A = \frac{ane\sin E_k}{1 - e\cos E_k} + \frac{2n\sqrt{1-e^2}}{(1 - e\cos E_k)^2}\times \times(C_{rs}\cos 2\phi_k - C_{rc}\sin 2\phi_k)\,, \quad (4)$$

$$B = \frac{an^2 e}{2}\frac{\cos E_k}{(1 - e\cos E_k)^3} - \frac{2n^2\sqrt{1-e^2}}{(1 - e\cos E_k)^4}\times \times\left[\sqrt{1-e^2}(C_{rs}\sin 2\phi_k + C_{rc}\cos 2\phi_k) + e\sin E_k(C_{rs}\cos 2\phi_k - C_{rc}\sin 2\phi_k)\right], \quad (5)$$

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$$C=\frac{an^3e}{6}\left[\frac{\sin E_k}{(1-\cos E_k)^4}+\frac{3e\sin E_k(\cos E_k-e)}{(1-\cos E_k)^5}\right]+$$
$$+\frac{2n^3\sqrt{1-e^2}}{3(1-\cos E_k)^6}[6e\sqrt{1-e^2}\sin E_k(C_{rs}\sin 2\phi_k+ \qquad . (6)$$
$$+C_{rc}\cos 2\phi_k)+(C_{rs}\cos 2\phi_k+C_{rc}\sin 2\phi_k)\times$$
$$\times(4e^2\sin^2 E_k-2\sqrt{1-e^2}-e\cos E_k+e^2\cos^2 E_k)]$$

In case of linear approximation the third and fourth terms in equation (3) are rejected. It is known that the error of approximation, when Taylor's series is used, does not exceed value of first rejected term [7]. Consequently the error of linear radius approximation is given by:

$$\Delta_{apr1}\le B(t-t_n)^2 . \qquad (7)$$

In case of quadratic approximation the fourth term in equation (3) is rejected. The error of quadratic radius approximation is given by:

$$\Delta_{apr2}\le C(t-t_n)^3 . \qquad (8)$$

Limited user range accuracy (URA) is provided by data in ephemeris message. Nominal value of URA is in range between 2m and 4000m [2]. In normal operational conditions this value is between 2m and 8m. Information for current URA value is also transmitted in navigation data.

If the errors of approximation (7) and (8) are of much more less value than URA, then they could be neglected. The conditions meeting these requirements are defined by:

$$\Delta_{apr1}\le 0.1\text{URA}; \qquad \Delta_{apr2}\le 0.1\text{URA} \qquad (9)$$

Consequently the errors of approximation should be bellow 0.2 m.

III. RESEARCHES

Researches about radiuses of all GPS satellite and corresponding approximation coefficients are made. The navigation data from satellites is collected for a 42 hours period using GPS receiver ANTARIS AEK-4P. Some of the results are shown below.

Studies show that the radius alteration and corresponding approximation coefficients are unique for each satellite. The alteration for satellite SV27 is the greatest, while for satellite SV17 it is the smallest. The similar results are obtained for approximation coefficients – their values for SV27 are the biggest, while for SV17 they are the smallest.

Fig. 1 illustrates radius of satellite SV 27 as function of time.

It is seen that the change of radius r_{sk} is periodic and the difference between minimum and maximum values is over 1000 km.

In Fig.2 approximation coefficients B and C for SV27 are shown. As it was mentioned the approximation accuracy depends on them.

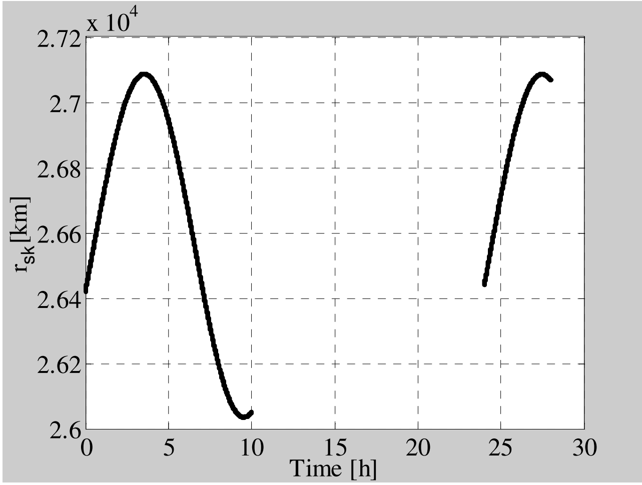

Fig. 1. Radius r_{sk} of SV27 as function of time.

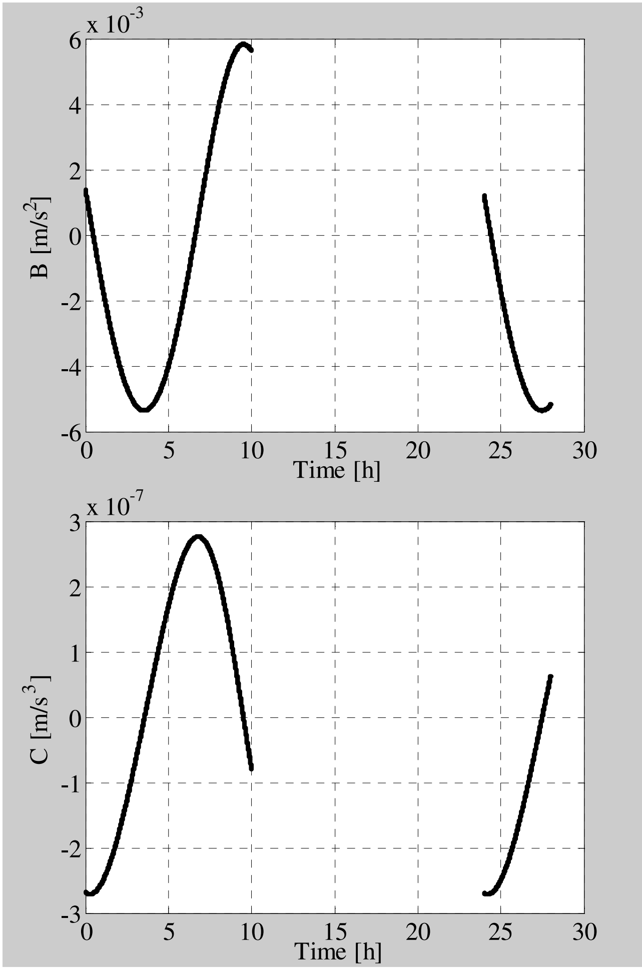

Fig. 2. Approximation Coefficients *B* and *C* for SV27.

As one can see there are periodical changes of coefficients values. The magnitude of B is up to 5.83×10^{-3} m/s^2 and the magnitude of C does not exceed 2.7643×10^{-7} m/s^3.

The radius of satellite SV17 is represented in Fig. 3. It is also periodic and the change of its value is less than 100 km.

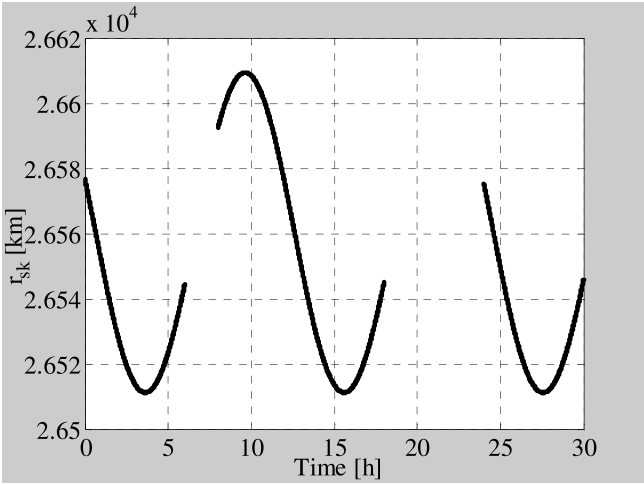

Fig. 3. Radius r_{sk} of SV17 as function of time.

Corresponding coefficients B and C for this satellite are shown in Fig. 4.

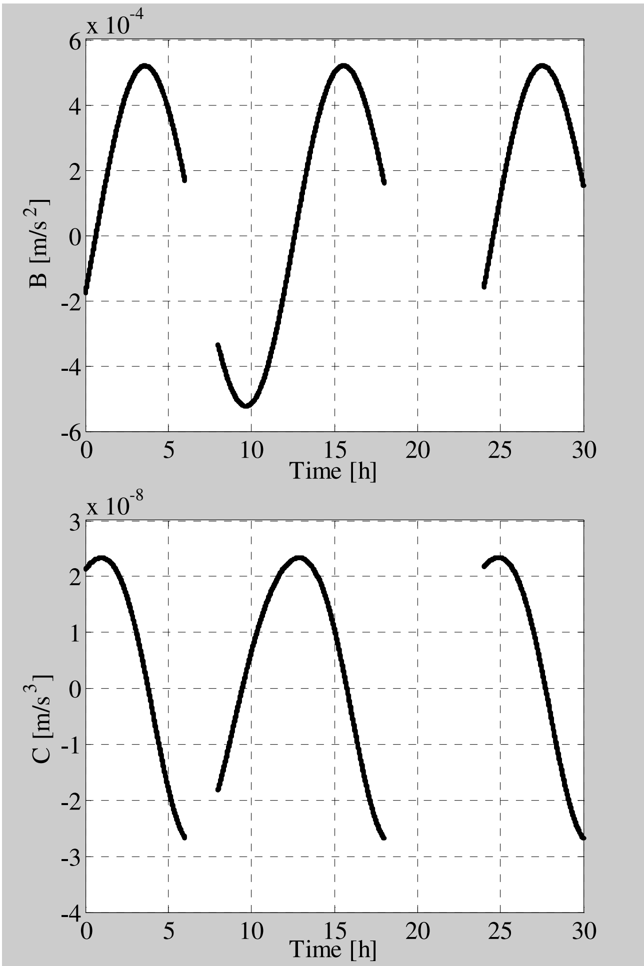

Fig. 4. Approximation Coefficients *B* and *C* for SV17.

Results indicate that the magnitude of B does not exceed 5.24×10^{-4} m/s^2, and the magnitude of C is up to 2.6867×10^{-8} m/s^3.

Analysis of results shows that there is a huge difference of radius alteration for these two satellites. This is because of the different orbital planes of two satellites. There is a difference in radius changes in all other satellites in constellation.

As it is seen in figures above there is absence of data for radius and respectively in calculated coefficients. This is because the satellite has not been visible from point of the GPS receiver reception at this time. Periodical behavior of the radius shows that the changes of its magnitude could not be extremely different.

The magnitudes of approximation coefficients B and C for SV27 are the worst case, because none coefficient for all other satellites exceeds their values. That's why these values are put in (7) and (8) to calculate the maximum of approximation errors Δ_{apr1} and Δ_{apr2}.

Next researches are focused to obtain the approximation errors as function of time interval of approximation (t-t_n).

The relation between Δ_{apr1} and the time interval, using linear approximation of the radius to both SV17 and SV27, is shown in Fig. 5.

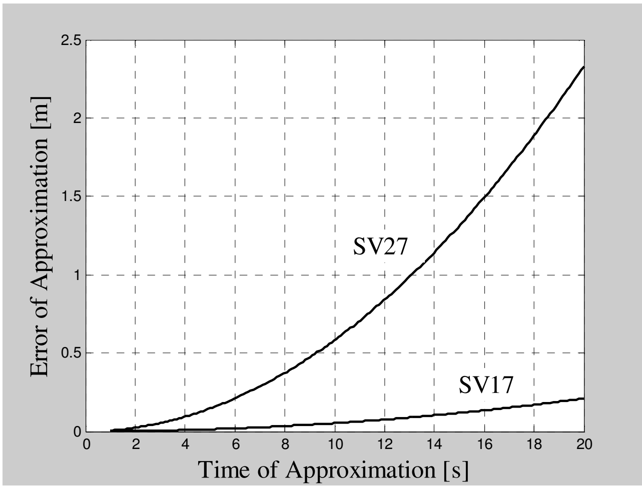

Fig. 5. Approximation error Δ_{apr1} for SV27 and SV17.

It is seen that Δ_{apr1} for SV27 is bigger than Δ_{apr1} for SV17 using the same time of approximation. Exact values of time interval, in case of maximum allowed error Δ_{apr1} = 0.2 m, are 5.8 seconds for SV27 and 19.5 seconds for SV 17. From these results it is obvious that the linear approximation is useable only if the rate of measurements is high enough. For instance, if the rate is 1 measurement per second, there are only five measured values in approximation time interval for satellite SV27.

Fig. 6 compares Δ_{apr2} for the same satellites, in case of quadratic radius approximation. As one can see the approximation time interval when Δ_{apr2} approach the value of 0.2 m, is larger than the time interval, when linear approximation is used. Exact values of time interval, in case of maximum allowed error, are 89 seconds for SV27 and 195 seconds (not shown in the Figure) for SV17.

Studies shows that the possible approximation intervals are large enough to use proposed approximation even when the rate of measurement is low. The GPS receiver used for researches outputs the user position coordinates four times per second. In this case 356 measurements are available for one approximation time interval in the worst case.

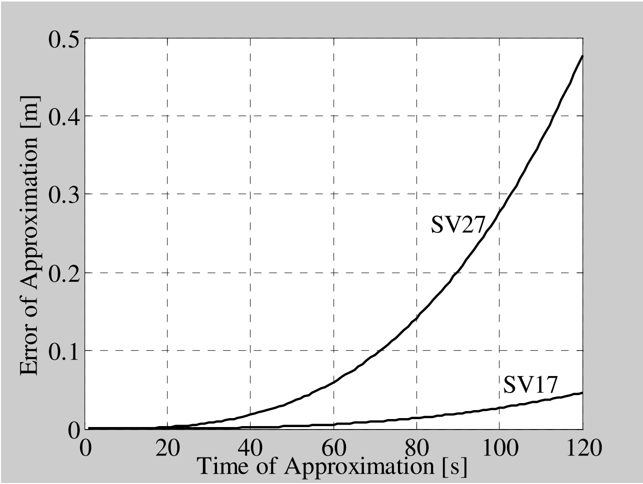

Fig. 6. Approximation error Δ_{apr2} for SV27 and SV17.

IV. CONCLUSION

The results of researches show that the radius alteration for every satellite in constellation could be approximated with quadratic or even linear function of time. This approximation can be used for future models.

The key features to approximate the radius alteration are the approximation errors and the approximation time interval. In case of using short time interval of approximation it is recommended to approximate radius with linear function of time.

When quadratic approximation is utilized longer time interval of approximation could be used , but the number of calculation operations will be increased because of additional coefficient B needed to compute, even though in this case, the number numerical calculation of Kepler's equation in standard algorithm [1, 2, 3] will be more seldom.

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