Studies of an Approximation of User-to-Satellite Range in GPS

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Abstract – **This paper presents some studies of an approximation of user-to-satellite range in GPS. Studies are made, using real data from satellites navigation message. The results for the best and the worst case are shown and the errors of approximations are obtained.**

Keywords – **GPS, approximation, user-to-satellite range.**

I. INTRODUCTION

Different methods, as Kalman filtering, the least square algorithms and etc., are used by GPS receivers to calculate user position. In these algorithms, models of various parameters are usually used. In the most of the used algorithms an approximation of user-to satellite range is utilized [4, 6, 7].

A user-to-satellite range approximation, using approximation of satellite radius proposed in [3] is presented in this paper. The studies are made to prove that the errors of proposed approximation are much smaller than errors in pseudorange measuring.

A U-blox GPS receiver ANTARIS AEK-4P is used to collect the necessary navigation data from satellites.

II. APPROXIMATION OF USER-TO-SATELLITE RANGE

Relation between radius of user R_u and distance user-to-k^{-th} satellite D_k is [8]

$$D_k = \sqrt{r_{sk}^2 + R_u^2 - 2r_{sk}R_u\cos\alpha_k} \tag{1}$$

where α_k is the angle between radius of user R_u and radius of k^{-th} satellite r_{sk}.

As was obtained in [3] the radius approximation for k^{-th} satellite is

$$r_{ak}(t) \approx r_{sk}(t_n) + A(t-t_n) + B(t-t_n)^2 + C(t-t_n)^3 \tag{2}$$

where A,B and C are approximation coefficients derived in [3]. If quadratic approximation is used only the first three terms are needed.

Substituting Eq. (2) into Eq. (1) yields:

$$\begin{aligned} D_{ak} = \{ & D_{k0}^2 + B^2(t-t_n)^4 + 2AB(t-t_n)^3 + \\ & + (A^2 + 2r_{sk0}B)(t-t_n)^2 + 2r_{sk0}A(t-t_n) - \\ & - 2R_u\cos\alpha_k[B(t-t_n)^2 + A(t-t_n)]\}^{\frac{1}{2}} \end{aligned} \tag{3}$$

where

$$D_{k0}^2 = r_{sk0}^2 + R_u^2 - 2r_{sk0}R_u\cos\alpha_k\,; \tag{4}$$

$$r_{sk0} = a(1 - e\cos E_k) + C_{rs}\sin(2\phi_k) + C_{rc}\cos(2\phi_k) \tag{5}$$

is the satellite radius, computed by standard algorithm from received ephemeris data [1, 2] at instant t=t_n; a is the semi major axis of satellite orbit, e is the eccentricity, E_k is the eccentric anomaly, C_{rs} and C_{rc} are sine and cosine corrections to orbital radius, ϕ_k is the argument of latitude.

The Eq. (3) could be rewritten in the following form:

$$D_{ak} = D_{k0}\sqrt{1+X}\,, \tag{6}$$

where

$$\begin{aligned} X = & \frac{B^2(t-t_n)^4 + 2AB(t-t_n)^3}{D_{k0}^2} + \\ & + \frac{(A^2 + 2r_{sk0}B)(t-t_n)^2 + 2r_{sk0}A(t-t_n)}{D_{k0}^2} - \\ & - \frac{2R_u\cos\alpha_k[B(t-t_n)^2 + A(t-t_n)]}{D_{k0}^2} \end{aligned} \tag{7}$$

It is useful to expand the root square in Eq. (6) in series to simplify expression. The condition that must be meet is X<<1 [8]. The range of distance D_{k0} is the widest in comparison with all other variables in Eq. (7). On the other hand distance D_{k0} is raised to the second power. That's why the influence of D_{k0} on X is the greatest. The worst case for approximation is when the value of D_{k0} is minimum. This takes place when the user is situated on the geodetic normal from the satellite position. The geometric relations for above situation are shown in Fig. 1.

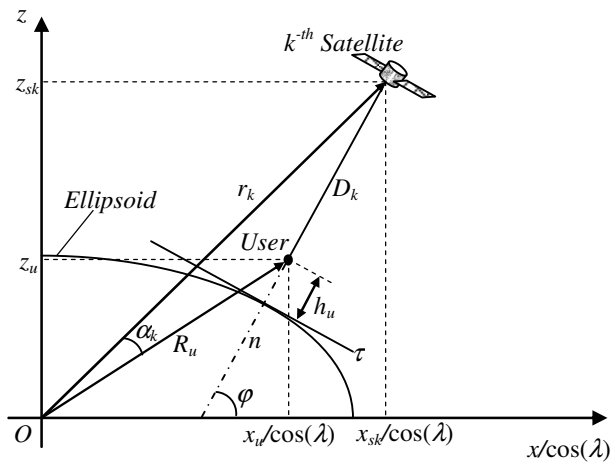

Fig. 1. Geometric relation

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The earth is represented as a reference ellipsoid according to WGS-84 [2]. In Fig. 1 O is the origin of both Earth-Centered Earth-Fixed (ECEF) and geodetic coordinate systems; φ and λ are corresponding geodetic latitude and longitude of satellite position; R_u is radius of user situated on the geodetic normal n; τ is tangent to the ellipsoid; h_u is user height above the ellipsoid.

Transformation from ECEF coordinates of k^{-th} satellite to corresponding geodetic coordinates is done to compute the minimum user-to-satellite range D_{k0}. From Fig.1 it is obvious that the geodetic height of satellite position is the distance from ellipsoid surface to the satellite. Maximum value of user height h_u must be subtracted from satellite height to obtain the minimum value of D_k. Usually the maximum value of user height h_u is 20 000 meters. Transformation from ECEF to Geodetic coordinate system (latitude φ, longitude λ and height h) are given by the following equations[5]:

$$\varphi = \tan^{-1}\left[\frac{z+\left[\left(\frac{a^2-b^2}{b^2}\right)b\left[\sin\left[\tan^{-1}\left[\frac{za}{b\sqrt{x^2+y^2}}\right]\right]\right]^3\right]}{\sqrt{x^2+y^2}-\left(\frac{a^2-b^2}{a^2}\right)a\left[\cos\left[\tan^{-1}\left[\frac{za}{b\sqrt{x^2+y^2}}\right]\right]\right]^3}\right], \quad (8)$$

$$\lambda = \begin{cases} \tan^{-1}\left(\frac{x}{y}\right), & x \geq 0 \\ 180^0 + \tan^{-1}\left(\frac{x}{y}\right), & x < 0, y \geq 0 \\ -180 + \tan^{-1}\left(\frac{x}{y}\right), & x < 0, y < 0 \end{cases}, \quad (9)$$

$$h = \frac{\sqrt{x^2+y^2}}{\cos(\varphi)} - \frac{a}{\sqrt{1-\left(\frac{a^2-b^2}{a^2}\right)\sin^2(\varphi)}}, \quad (10)$$

where a and b are the semi major and semi minor axis of the earth ellipsoid; x, y and z are ECEF coordinates.

Minimum values of D_{k0} are calculated using a real ephemeris data for each satellite, collected for 42 hours time of observation with U-blox GPS receiver ANTARIS AEK-4P. Using these data and Eqs. (8), (9), (10) the geodetic coordinates of satellites are computed. The ECEF coordinates of satellites x_{sk}, y_{sk}, z_{sk} are also calculated by ephemeris data, using standard algorithm [1, 2].

The ECEF coordinates of user could be calculated, using transformation equations from geodetic to ECEF coordinates given by [1, 5]:

$$x = \left[\frac{a}{\sqrt{1-\left(\frac{a^2-b^2}{a^2}\right)\sin^2(\varphi)}} + h\right]\cos(\varphi)\cos(\lambda) \quad (11)$$

$$y = \left[\frac{a}{\sqrt{1-\left(\frac{a^2-b^2}{a^2}\right)\sin^2(\varphi)}} + h\right]\cos(\varphi)\sin(\lambda) \quad (12)$$

$$z = \left[\frac{a}{\sqrt{1-\left(\frac{a^2-b^2}{a^2}\right)\sin^2(\varphi)}}\left[1-\left(\frac{a^2-b^2}{a^2}\right)\right] + h\right]\sin(\varphi) \quad (13)$$

User radius R_u is given by:

$$R_u = \sqrt{x_u^2 + y_u^2 + z_u^2}, \quad (14)$$

where x_u, y_u, z_u are ECEF coordinates of user position.

When the user is located on the geodetic normal of satellite the user longitude and latitude are coincided with the satellite longitude λ and latitude φ, and the user height h_u is considered to be 20 000 meters (the worst case).

The values of $\cos\alpha_k$ could be computed using the following expression [8]:

$$\cos\alpha_k = \frac{\vec{r}_{sk}\vec{R}_u}{r_{sk}R_u}, \quad (15)$$

where r_{sk} and R_u are the magnitudes of $\vec{r}_{sk}$ and $\vec{R}_u$.

From the above equation the product $R_u\cos\alpha_k$ could be derived as follows [8]:

$$R_u\cos\alpha_k = \frac{x_{sk}x_u + y_{sk}y_u + z_{sk}z_u}{r_{sk}}, \quad (16)$$

Eqs. (4), (5), (14) and (16) allow the values of X to be calculated, in the worst case for each satellite.

III. Researches

Various time intervals of approximation (t-t_n) are used to compute the values of X. Studies show that the values of X are unique for each satellite. They are the greatest for satellite SV27, while for satellite SV17 they are the smallest. So the worst case of approximation takes place for satellite SV27. In [3] is shown that the maximum possible time interval of satellite radius approximation, using Eq. (2) for SV27, is 89 seconds. This determines the maximum allowed time interval of

user-to-satellite range approximation. The studies are made for time intervals from 10 to 90 seconds with a step of 10 seconds.

In Figs. 2 and 3 the values of X for satellite SV27 are illustrated. In Fig. 2 the results for 20 second time interval are shown. In Fig. 3 the results for 90 second time interval are presented.

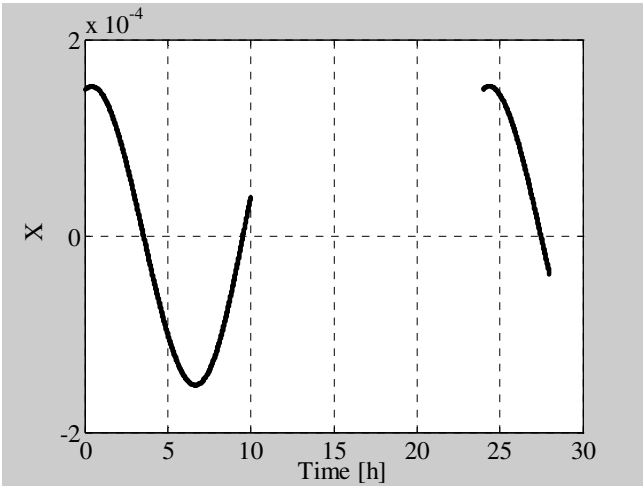

Fig. 2. Values of X in case of time interval of approximation 20 seconds

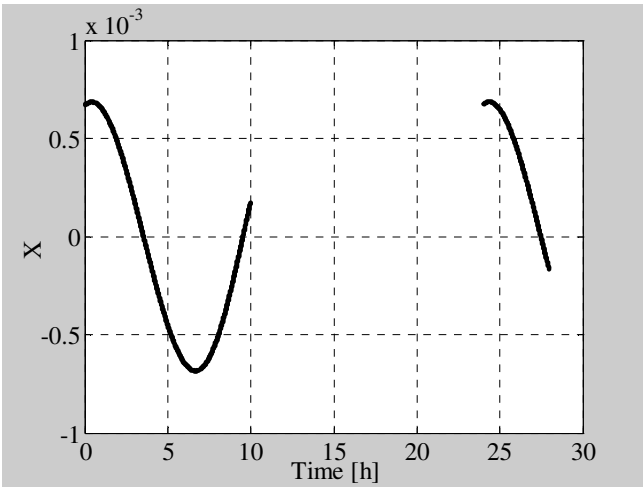

Fig. 3. Values of X in case of time interval of approximation 90 seconds

As it is seen in above figures there is absence of data for X. This is because the satellite has not been visible from the point of the GPS receiver at this time.

It is obvious that the values of X do not exceed 1×10^{-3} which is small enough to expand user-to-satellite range in series [8]:

$$D_{ak1} \approx D_{k0}\left(1+\frac{X}{2}-\frac{X^2}{6}+\frac{X^3}{48}-...\right) \tag{17}$$

In order to simplify (17) the non-linear terms are rejected. In this case the final result of user-to-satellite range approximation is obtained as:

$$\begin{aligned} D_{ak1} \approx D_{k0}\left(1+\frac{X}{2}\right) = D_{k0} + \\ + \frac{B^2(t-t_n)^4 + 2AB(t-t_n)^3}{2D_{k0}} + \\ + \frac{(A^2+2r_{sk0}B)(t-t_n)^2 + 2r_{sk0}A(t-t_n)}{2D_{k0}} - \\ - \frac{2R_u\cos\alpha_k[B(t-t_n)^2 + A(t-t_n)]}{2D_{k0}} \end{aligned} . \tag{18}$$

It is known that the user–equivalent range error UERE budget depends on the space segment, control segment, and user segment sources [1]. UERE is 6.6 meters (1 sigma) for Precise Point Service (PPS), and 8 meters (1 sigma) for Standard Positioning Service (SPS) [1]. The errors due to ionospheric and tropospheric delays can be reduced at least with 50%, using the corresponding model [2], which parameters are transmitted in navigation data by every satellite. Thus the UERE could be decreased to 3 meters (1 sigma) [1, 2].

If the error of approximation

$$\Delta D_{apr} = D_{ak} - D_{ak1} \tag{19}$$

is of much more less value than UERE, then it could be neglected. The conditions meeting these requirements are defined by:

$$\Delta D_{apr} \le 0.1UERE \tag{20}$$

Consequently the error of approximation should be bellow 0.3 meters.

Researches about the values of error approximation ΔD_{apr} for satellite SV27 (the worst case) are made for time approximation intervals from 10 to 90 seconds with a step of 10 seconds. Some of the results are shown in Fig. 4 and Fig. 5.

In Fig. 4 errors of approximation, when time interval of approximation is 60 seconds, as a function of observation time, are presented.

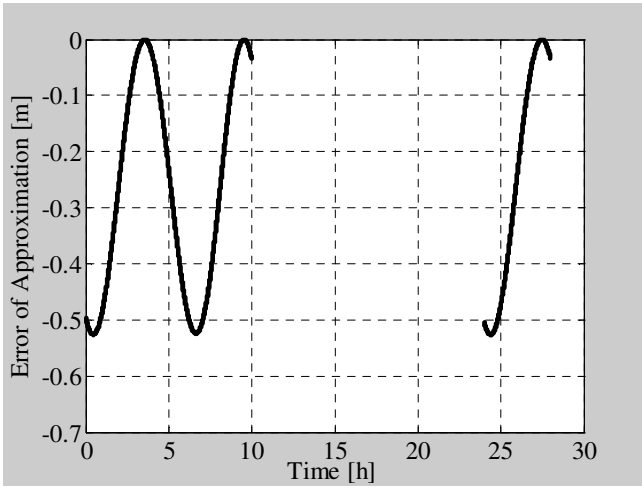

Fig. 4. Error of approximation in case of time interval of approximation 60 seconds

As it can be seen from Fig. 4 the maximum value of error is over 0.5 meters. This does not agree with Eq. (20) and consequently the time interval of approximation must be shorter.

In Fig. 5 errors of approximation, when time interval of approximation is 40 seconds, as a function of observation time, are presented.

Fig. 5 illustrates that the maximum value of error does not exceed 0.25 meters. This is in agreement with Eq. (20) and consequently the time interval of approximation 40 seconds could be used and even longer one could be used.

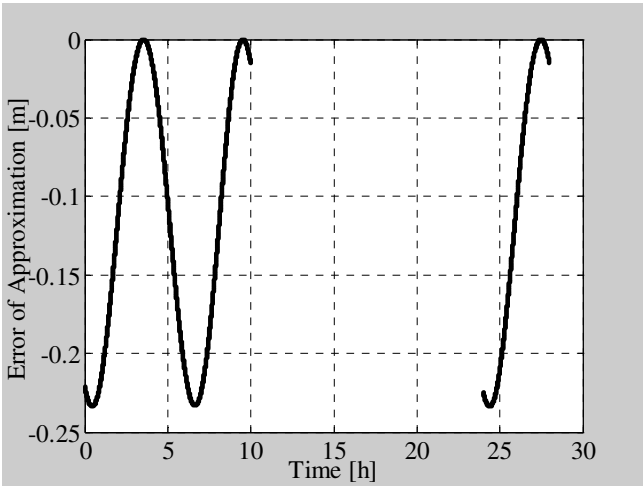

Fig. 5. Error of approximation in case of time interval of approximation 40 seconds

The results prove that the requirements to time interval of user-to-satellite range approximation are much stronger than the time interval of satellite radius approximation obtained in [3]. Hence the time interval of approximation must be chosen according to Eq. (20).

In Fig. 6 the results from studies of the error of approximation as a function of time interval of approximation are shown. The errors of approximation for the best (satellite SV17) and the worst (satellite SV27) case are presented.

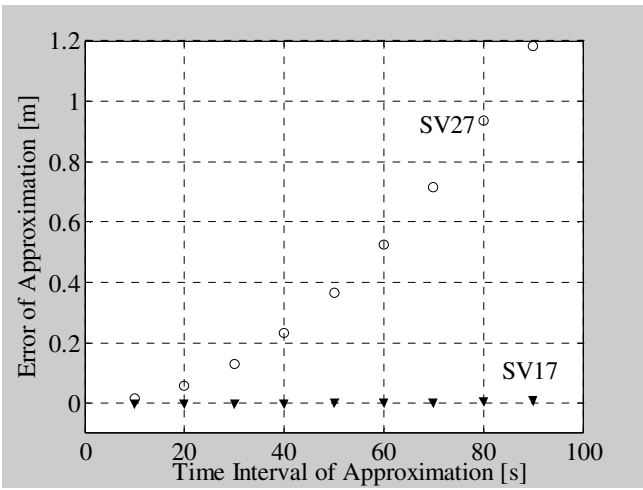

Fig. 6. Error of approximation for satellites SV27 and SV17.

As one can see in case of approximation for satellite SV17 the longer time interval of approximation could be used, because even for 90 seconds time interval the approximation error ΔD_{apr} is 0.0107 meters.

In the same time interval of approximation the approximation error ΔD_{apr} for satellite SV27 is 1.1839 meters, which goes far beyond the error bounder.

IV. Conclusion

Proposed approximation represents user-to-satellite range as direct function of time. Eqs. (16) and (18) show that this approximation can be used for modeling the measured pseudoranges in the way that allows using kalman filtering or least square algorithm. The obtained results indicate that the requirements to user-to-satellite range approximation are much stronger than to satellite radius approximation. Despite of this for one satellite radius approximation interval several time intervals of user-to-satellite range approximation could be used.

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