# Common theory, approximation method and design of electrical filters based on Hausdorff polynomials

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*Abstract* - **An application of Hausdorff approximation in the modern theory of electrical filters design is shown in the paper. A translated Hausdorff polynomial is defined, which leads to realizable transmission functions of electrical filters. Transmission functions of Hausdorff low-pass filter-prototype and two types of inverse Hausdorff filters are shown. Their frequency characteristics are analyzed.**

*Keywords* - **Approximation, Polynomial, Design, Inverse filter, Hausdorff, Chebyshev, Magnitude response.**

## I. Introduction

In modern filter theory the synthesis is performed by appropriate characteristic function (Fig.1) approximation.

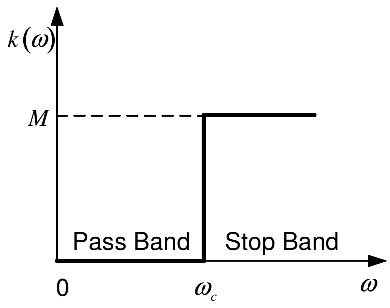


Fig. 1. Characteristic function

The most popular approximations are the canonical – equal-ripple in pass-band (Chebyshev I, Cauer) and maximum-flat in pass-band (Butterworth, Hourglass, Chebyshev II). Another approximations with intermediate properties also exist. Such approximations are of Bessel, Legendre and Gauss.

On Fig. 2 and Fig. 3 amplitude responses and phase responses of filters obtained by mentioned approximations are shown.

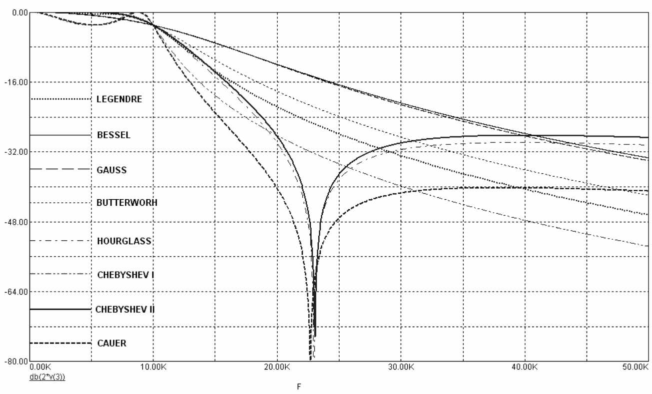


Fig. 2. Comparison of magnitude responses [dB]

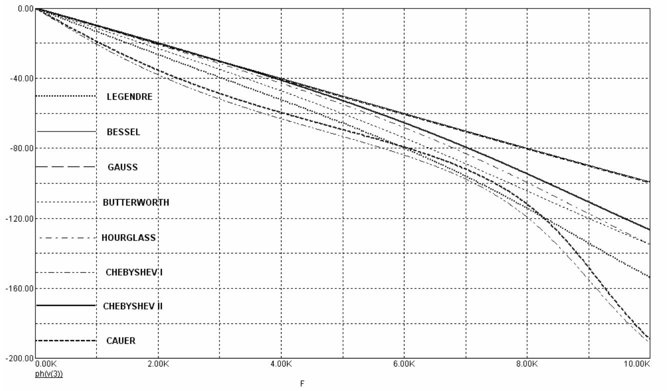


Fig. 3. Comparison of phase responses [deg]

As can be seen, the filters, as linear electrical circuits, have contradictory behaviour. Those with better selectivity (Chebyshev I and Cauer) have non-linear phase-responses and vice versa - those with worse selectivity (Gauss, Bessel, Butterworth) have more linear phase responses. The Chebyshev's II filters combine better selectivity and acceptable linearity of phase response.

The advance of the digitalization in the last years is a motivation for finding new approximations, which lead to decreased distortions of digital signal processing. A step in this direction is the approximation in Hausdorff metric.

## II. Approximation in Hausdorff metric. Formation of transmission function of Hausdorff low-pas filter prototype

The Bulgarian Academy of Science (BAS) conducted a research of $\varepsilon$-entropy of the space, a new approximation in Hausdorff metric [5] is offered in the eighties. The theoretical results of this work were successfully applied in an antenna array design for the Technical University of Sofia [2] and in the transmission functions direct synthesis of digital filters in BAS [3]. In the theory, an algebraic polynomial accomplishing the best approximation of "shifted" delta function in Hausdorff metric is offered

$$P_n(x) = \varepsilon T_n\left(\frac{2x+\alpha\varepsilon}{2-\alpha\varepsilon}\right), \tag{1}$$

where $\varepsilon$ is Hausdorff distance, $T_n$ is Chebyshev's polynomial of first kind and $n$ degree; $\alpha$ is a parameter, and the factor (product) $\alpha\varepsilon$ determines the bandwidth in which the polynomial approximates the "shifted" delta function in point

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1, where it has infinitive steepness (Fig. 4). The relations between polynomial parameters [1] are defined by equation

$$\alpha\varepsilon = 2\frac{\mathrm{ch}\left[\dfrac{1}{n}\mathrm{Ach}\left(\dfrac{1}{\varepsilon}\right)\right]-1}{\mathrm{ch}\left[\dfrac{1}{n}\mathrm{Ach}\left(\dfrac{1}{\varepsilon}\right)\right]+1}. \qquad (2)$$

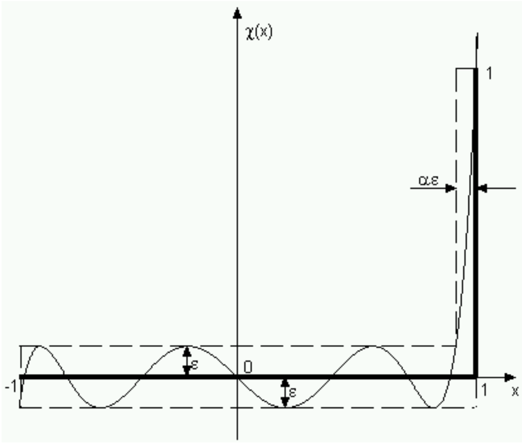


Fig. 4. Approximation with Hausdorff polynomial

As seen on the figure the Hausdorff polynomial approximates function with outline of filter characteristic function (Fig. 1). The pass-band is determined in the interval $[0,1-\alpha\varepsilon]$, stop-band in the interval $(1-\alpha\varepsilon,\infty)$. Considering Eq. (1) and Fig.4 in the defined area the following inequalities (InEq) are valid:

$$0<\varepsilon<1;\ 0<\alpha\varepsilon<1. \qquad (3)$$

The representation of the polynomial as a rational function of its argument can be determined by the coefficients of the Chebyshev's polynomial. A fifth-order Hausdorff polynomial will look like

$$P_5(x)=\varepsilon\left[16\left(\frac{2x+\alpha\varepsilon}{2-\alpha\varepsilon}\right)^5-20\left(\frac{2x+\alpha\varepsilon}{2-\alpha\varepsilon}\right)^3+5\frac{2x+\alpha\varepsilon}{2-\alpha\varepsilon}\right]. \qquad (4)$$

Fig. 5 shows the graphical representation of the polynomial with values of $\varepsilon=0.07556$ and $\alpha\varepsilon=0.2$.

It can be seen in Eq. (4) that after $\varepsilon$ and $\alpha\varepsilon$ are substituted with their values, raising to the power, removing the brackets and reducing, the Hausdorff polynomial can be represented as a rational function of its argument.

$$P_5(x)=2.04738\,x^5+1.02369x^4-1.86824x^3-$$
$$-0.601419x^2+0.358612x+0.0399253. \qquad (5)$$

It is known from the theory that the transmission function of a low-pass filter is a fraction-rational. To achieve a realizable transmission function, the polynomial in the denominator is required to be a strict Hurwitz' polynomial. The formation of the transmission function, using the polynomial in Eq. (4), cannot lead to a realizable transmission function for every $\alpha\varepsilon$ [1]. That is because the Hausdorff polynomial in Eq. (4) is neither even nor odd function of its argument, which can be seen from Eq. (5) and Fig. 5. The polynomial graph is asymmetrical with regards the centre of the coordinate system.

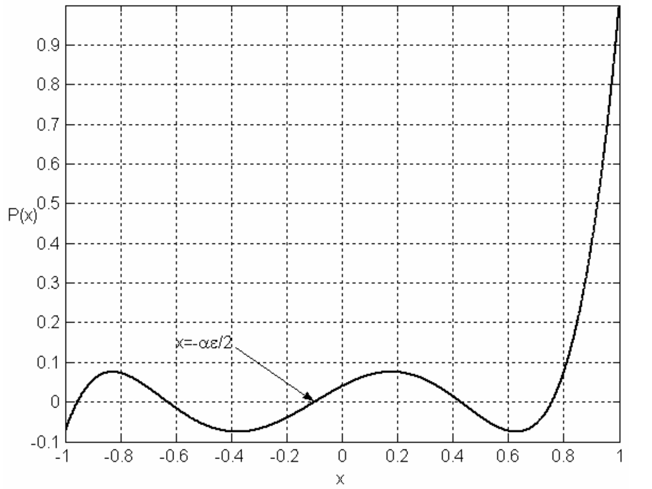


Fig. 5. Hausdorff polynomial, $n$=5, $\varepsilon$ =0.07556, $\alpha\varepsilon$ =0.2

The investigations showed, that for the value of argument $x=-\alpha\varepsilon/2$, the Hausdorff polynomials have a typical point: a root, when the polynomial is even (Fig.5) and a local extremum, when it is odd. If a translation with a value of $\alpha\varepsilon/2$ in the positive direction is performed, the translated polynomials lead to realizable transmission functions.

$$P_n^T(x)=\varepsilon T_n\left(\frac{2x}{2-\alpha\varepsilon}\right)=\varepsilon T_n\left(\frac{x}{1-\alpha\varepsilon/2}\right). \qquad (6)$$

It is proved in [1], that the translated polynomial represents the only and the best approximation of "shifted" delta function, translated with $\alpha\varepsilon/2$ in positive direction, in Hausdorff metric.

The square of the transmission function module of low-pass Hausdorff filter-prototype has the form

$$|A_n|^2(\omega)=\frac{1}{1+\varepsilon^2T_n^2\left(\dfrac{2\omega}{2-\alpha\varepsilon}\right)}=\frac{1}{1+\varepsilon^2T_n^2\left(\dfrac{\omega}{1-\alpha\varepsilon/2}\right)}, \qquad (7)$$

where $\omega$ is the angle frequency.

## III. Design of Hausdorff filters

It is known that the module of the transmission function defines the magnitude response of the filter. In Eq. (7) the argument of the Chebyshev's polynomial is divided by the expression $(1-\alpha\varepsilon/2)$. Considering InEq. (3), the expression is positive number less than 1. That leads to a scale "shrinking" of the magnitude response compared to the magnitude response of a Chebyshev's I filter as shown on Fig. 6.

The low-pass Hausdorff filter-prototype design led to the following results:

- The pass-band of the Hausdorff filter-prototype is "shrunk" by coefficient $(1-\alpha\varepsilon/2)$.

• For the same order and ripple in the pass-band, the magnitude response of a low-pass Hausdorff filter-prototype has the same steepness and attenuation in the stop-band as a Chebyshev's I filter.

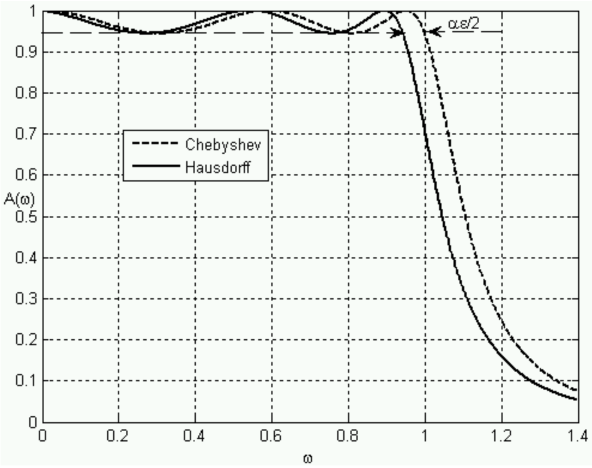


Fig. 6. Magnitude responses of Hausdorff and Chebyshev's I filters

• Hausdorff low-pass filter-prototype has the same linearity in phase response in the pass-band as a Chebyshev's I filter.

• Hausdorff low-pass filter-prototype has the same evenness of GDT in the pass-band as a Chebyshev's I filter.

• Hausdorff low-pass filter-prototype cannot have bigger unevenness in the pass - band than $1/\sqrt{2}$ (-3.01dB), because $\varepsilon < 1$ InEq. (3).

• Poles' quality factors have the same values as a Chebyshev's I filter.

From the above mentioned, a conclusion could be made: *The Hausdorff low-pass filter-prototype cannot find practical application, because of the undesirable "shrinking" of the pass-band.*

The inverse Hausdorff filters are more interesting from a design point of view. Two types are described here – A and B as explained next. The modules of their transmission functions are shown with the following two equations:

$$\left|A_n(\omega)\right|_A = \sqrt{\frac{\varepsilon^2 T_n^2\left[\dfrac{1}{\omega(1-\alpha\varepsilon/2)}\right]}{1+\varepsilon^2 T_n^2\left[\dfrac{1}{\omega(1-\alpha\varepsilon/2)}\right]}}; \tag{8}$$

$$\left|A_n(\omega)\right|_B = \sqrt{\frac{\varepsilon^2 T_n^2\left(\dfrac{1-\alpha\varepsilon/2}{\omega}\right)}{1+\varepsilon^2 T_n^2\left(\dfrac{1-\alpha\varepsilon/2}{\omega}\right)}}. \tag{9}$$

The difference is that for type A the expression $(1-\alpha\varepsilon/2)$ multiplies the argument of Chebyshev's polynomial while for type B it divides the argument. That naturally leads to a scale expanding/shrinking of the magnitude response such as for the low-pass filter-prototype. In contrast to that, the inverse Hausdorff filters have magnitude response poles – real frequencies, where the attenuation is infinitive. This allows the method of concluded equal-ripple approximation [4], [6] to be applied to these filters. Inverse Hausdorff filters could be designed to keep their cut-off frequency, and frequencies in the stop-band could be proportional to the $\alpha\varepsilon$ factor.

In the synthesis the transfer function is represented as relation of three polynomials *e(s)*, *p(s)* and *q(s)* of complex frequency $s = j\omega$. The polynomial *e(s)* is Hurwitz strict polynomial and its zeros $\omega_i$ represent the filter own frequencies and these of *p(s)* - the extreme frequencies $\omega_{\infty i}$, for which the transfer function has infinite attenuation. Calculating two of polynomials usually solves the synthesis task and the third is defined by the equation:

$$e(s)e(-s) = p(s)p(-s) + q(s)q(-s) \tag{10}$$

The zeros of *e(s)* and *p(s)* can be found as follows:
For A-type:

$$\omega_i = \frac{1}{\left(1-\dfrac{\alpha\varepsilon}{2}\right)(\sigma_i + j\Omega_i)}; \tag{11}$$

$$\omega_{\infty i} = \frac{j}{\left(1-\dfrac{\alpha\varepsilon}{2}\right)\cos\left(\dfrac{2i-1}{n}\dfrac{\pi}{2}\right)}. \tag{12}$$

For B-type:

$$\omega_i = \frac{1-\dfrac{\alpha\varepsilon}{2}}{\sigma_i + j\Omega_i}; \tag{13}$$

$$\omega_{\infty i} = \frac{j\left(1-\dfrac{\alpha\varepsilon}{2}\right)}{\cos\left(\dfrac{2i-1}{n}\dfrac{\pi}{2}\right)}, \tag{14}$$

where:

$$\sigma_i = -\sin\left(\frac{2i-1}{n}\frac{\pi}{2}\right)\mathrm{sh}\left[\frac{1}{n}\mathrm{Ash}\left(\frac{1}{\varepsilon}\right)\right]; \tag{15}$$

$$\Omega_i = \cos\left(\frac{2i-1}{n}\frac{\pi}{2}\right)\mathrm{ch}\left[\frac{1}{n}\mathrm{Ash}\left(\frac{1}{\varepsilon}\right)\right]. \tag{16}$$

Fig. 7 compares the magnitude responses of the two types inverse Hausdorff filters with Chebyshev's II filter.

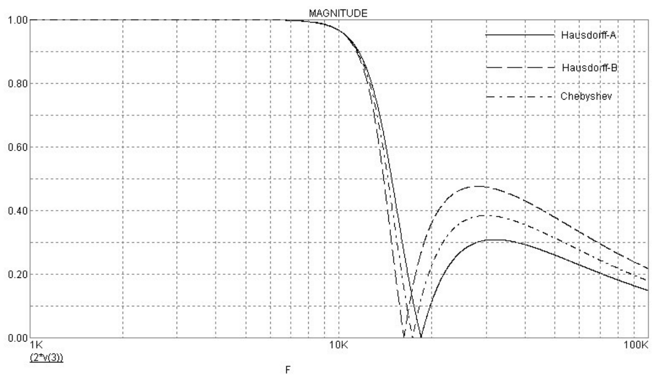


Fig. 7 Comparison of magnitude responses

It can be seen from the figure, that the inverse Hausdorff filters type A (IHF-A) have less magnitude response steepness in the area between the cut-off frequency and the frequency of an infinite attenuation and higher attenuation in the stop-band in comparison with Chebyshev's II filter. As for IHF-B it is the contrary – a bigger steepness and less attenuation in the stop-band.

Fig.8 compares the phase responses. It is seen from the comparison that IHF-A has the best linearity. In this case improvement of the linearity compared with Chebyshev's II filter for the frequency 6kHz is 5.3%. IHF-B has the worse linearity.

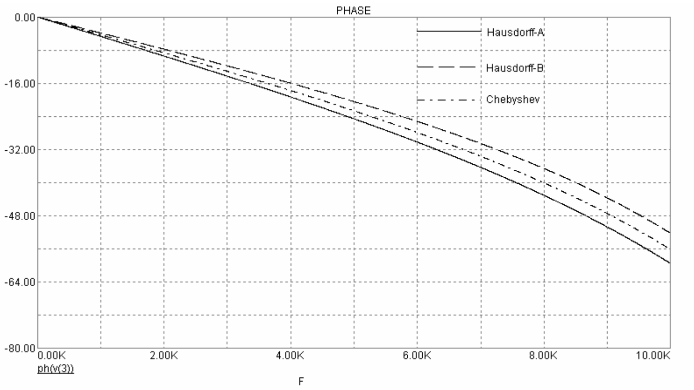


Fig. 8. Comparison of phase responses

Fig. 9 compares the GDT of the filters.

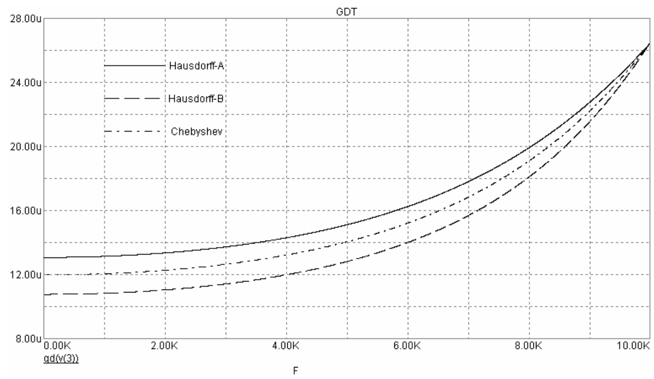


Fig. 9. Comparison of GDT

The improvement of the evenness of GDT of IHF-A compared with Chebyshev's II filter in this case is a little more than 9%. GDT of IHF-B is the most uneven.

Fig.10 compares the poles of the filters in a complex domain.

The poles values distribution could be used to judge the poles' quality factors of the filters. A criterion is the remoteness of the poles with imaginary part from the imaginary axis. The poles of IHF-A are the most remote ones. That means they will have the lowest values of the poles' quality factors and they will have easier values for components realization.

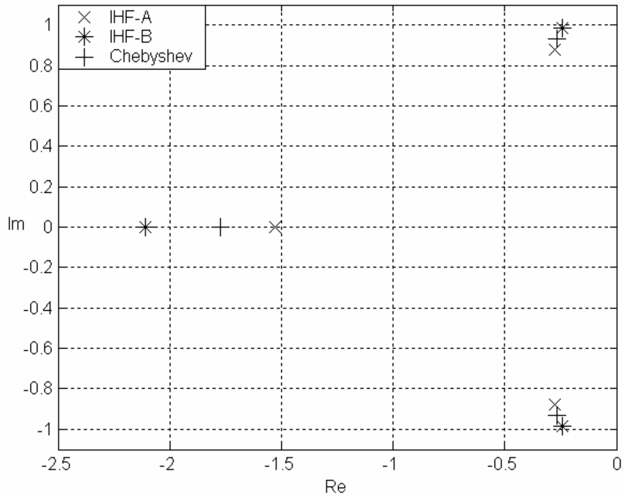


Fig. 10. Poles values distribution

## IV. CONCLUSION

The approximation implementation in Hausdorff metric doesn't lead to a "revolution" in the filter design domain. But it could be said, that it is an "evolution" in this domain of investigation. The implementation of the Chebyshev's polynomial in Eq.1, creates filters with characteristics similar to Chebyshev's filters. The difference is defined by the value of the factor $\alpha\varepsilon$.

Low-pass Hausdorff filter cannot find practical application, because there is undesirable "shrinking" of the pass-band, proportional to the factor $\alpha\varepsilon$.

When a suppression of the signals close to the cut-off frequency is required, the IHF-B filters can be used due to their bigger steepness of the magnitude response. They have more nonlinear phase response, more uneven GDT and higher values of poles' quality factors

IHF-A filters are probably the best that can be achieved when applying the Hausdorff metric approximation. In comparison with Chebyshev's II filters they have higher attenuation in the pass–band, more linear phase response, more even GDT and lower poles quality factors. These properties make them appropriate for implementing in filtration of digital signals.

## V. REFERENCES


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