Sets of Representative Points and Modified Hausdorff Distance for Image Registration

Yulka P. Petkova[1], Mariana Ts. Stoeva[2]

Abstract – **The present work is aroused from the striving after decreasing the computational complexity of the similarity measures, which are used in the tasks for searching and localizing the before known template into the larger image, keeping the accuracy of the measure. We propose criteria for forming the sets of representative points and a modification of the classical Hausdorff distance, which uses the specific features of these sets. We theoretically prove, that the modified distance has smaller computational complexity than classical.**

Keywords – **Hausdorff distance, Template matching, Representative points, Image registration.**

I. Introduction

Systems for pattern recognition and image processing have great applications in many fields as for example: character recognition, scene analysis, analysis of medical signals and images, person identification, face recognition and also in the robot technique in different areas of industry. One of the most frequently solved tasks in these systems is the task of searching, registration and localizing a properly chosen template into a larger image. This task is being solved using different methods for coinciding properly chosen sets of representative points and using different distances to evaluate matching between them. The main goal of the improvements of these methods is to increase or to keep their reliability and to decrease their computational complexity as well.

In the literature there are many classifications of registration methods. One of them is for example that, proposed in [1]. According to this classification, methods for image registration can be divided into the following classes:

- methods, which directly use the intensity value in each pixel, i.e. correlation methods;
- methods, which use the transformation in the frequency domain, i.e. methods, based on the Fast Fourier Transformation;
- methods, which use the representation of the template with so called low level features, for example – edges, corners, and contours, these are so called “feature based” methods;
- methods, which use high level characteristics, like identified object (or parts of objects) or relevancies between their features; these are so called graph-theoretic methods.

[1] Yulka P. Petkova is with the Department of Computer Science and Technologies, Technical University of Varna, 1, Studentska Str., Varna 9010, Bulgaria, Email: jppet@mbox.digsys.bg

[2] Mariana Ts. Stoeva is with the Department of Computer Science and Technologies, Technical University of Varna, 1, Studentska Str., Varna 9010, Bulgaria

Each of these methods can be realized in different way, using different similarity (or matching) measures. The most successful matching measures are based on the distance transformations, because of their high stability to missing or partly occluded data [2], [3], [4]. In these methods binary templates represent objects. Most frequently these template contain information about the coordinates of the representative points. Hausdorff distance is one of the measures, which determine the distance between two sets of points. It measures the extent to which each point of a template set lies near some point of an image set and vice versa.

Most frequently sets of representative points are formed from the edge points, from the corner points or from the contour points, because these points bring the biggest information about the image.

Our goal in this paper is directed to the forming of the sets of representative points and also to a modification of the Hausdorff distance in order to decrease its computational complexity, using the features of the point sets and keeping the worth of the measure.

II. Forming the Set of Representative Points

A. Edge Points

In [7] we propose a new edge definition, based on the interrupted first derivative of intensity function. In practice at the points of interruption second derivative has local extreme values. The rule for determining if a point belongs to an edge (for one-dimensional case) is:

$$\text{If} \quad \left| \underset{i}{extr}\left(\frac{d^2 I(x)}{dx^2}\right) \right| > \theta, \quad \text{then} \quad i \in E, \tag{1}$$

where $I(x)$ is the intensity function; E is a set of points i, which are edge points; θ is a before settled threshold.

Here we propose to divide such formed set E into two subsets, depending on the sign of the local extremum. We propose the following **definition:**

Definition 1: The set E consists of two subsets, which are formed according the following criteria:

$$\text{If} \quad \left(\underset{i}{extr}\left(\frac{d^2 I(x)}{dx^2}\right) \right) > 0, \quad \text{then} \quad i \in E_L \tag{2.1}$$

and

$$\text{If} \quad \left(\underset{i}{extr}\left(\frac{d^2 I(x)}{dx^2}\right) \right) < 0, \quad \text{then} \quad i \in E_H \tag{2.2}$$

The set of all edge points E is a sum of both subsets

$$E = E_L \cup E_H \ . \tag{3}$$

B. Informative points, selected with the method of equipotential planes

In [6] we propose extracting the representative points to be conformed to the criterion of D-optimality. According to this criterion the most informative points lie on the protruded peripheral wrapping of the object [5]. Thus, we propose the choice to be made by equipotential planes, which are parallel to the plane *xOy* and which cut the three-dimensional image profile (relief) on proper intensity levels. Thus, the extracted points outline the horizontal contours of the local "hollows" and "hills" from the three-dimensional image profile.

Here we propose to divide such formed sets into two subsets, depending on their belonging to local "hollow" or "hill", according to the following **definition:**

Definition 2: Set of points EP consists of two subsets, which are formed according to the following criteria:

$$\text{If}\begin{pmatrix} [(PI(i,j) \le Pt_{max}) \cap (PI(i+1,j) > Pt_{max})] \cup \\ \cup [(PI(i,j) \ge Pt_{max}) \cap (PI(i+1,j) < Pt_{max})] \end{pmatrix}, \text{then} (i,j) \in EP_H \tag{4.1}$$

$$\text{If}\begin{pmatrix} [(PI(i,j) \le Pt_{min}) \cap (PI(i+1,j) > Pt_{min})] \cup \\ \cup [(PI(i,j) \ge Pt_{min}) \cap (PI(i+1,j) < Pt_{min})] \end{pmatrix}, \text{then} (i,j) \in EP_L \tag{4.2}$$

where: $PI(i,j)$ is the intensity at the point (i,j); Pt_{min}, Pt_{max} are intensity values, which determine the distance between the *xOy* plane and the planes, cutting low and high parts of the relief ("hollows" and "hills"); EP_L, EP_H are sets of points, outlining "hollows" and "hills".

The set of all informative points EP is an union of both subsets:

$$EP = EP_L \cup EP_H \tag{5}$$

Both subsets can be additionally enriched in addition with the points, which describe "ridges" and "tablelands" in the separated "hills" and "valleys" and "lowlands" in the separated "hollows", as we propose in [8].

As it is seen both sets of informative points (E and EP) are composed of two subsets, which are formed according to the criteria (2.1) and (2.2) or (4.1) and (4.2). The idea, which arises here, is to use this property of the sets when we use the Hausdorff distance for determining the distance between them.

III. MODIFIED HAUSDORFF DISTANCE

The classic Hausdorff distance is defined [4] as follows:

$$H(A,B) = max(h(A,B),\ h(B,A)) \tag{6}$$

where $A = \{a_1, a_2, \dots a_p\}$ and $B = \{b_1, b_2, \dots b_q\}$ are finite, not empty sets of points, i.e. $p, q \ne 0$, and $h(A,B)$ and $h(B,A)$ are the directed distances.

Let the compared sets of points consist of two subsets, which are obtained using ones and the same criteria, particularly these, proposed in the previous section:

$$A = A_1 + A_2\ ;\ A_1 = \{a_{11}, a_{12}, \dots, a_{1p_1}\};\ A_2 = \{a_{21}, a_{22}, \dots, a_{2p_2}\}$$

$$\text{where } p_1 + p_2 = p \text{ and } p_1, p_2 \ne 0 \tag{7.1}$$

$$B = B_1 + B_2\ ;\ B_1 = \{b_{11}, b_{12}, \dots, b_{1q_1}\};\ B_2 = \{b_{21}, b_{22}, \dots, b_{2q_2}\}$$

$$\text{where } q_1 + q_2 = q \text{ and } q_1, q_2 \ne 0 \tag{7.2}$$

Then the equation (6) can be written as:

$$\begin{aligned} H(A,B) &= H(A_1 + A_2, B_1 + B_2) = max(h(A,B),\ h(B,A)) = \\ &= max(h(A_1 + A_2), h(B_1 + B_2)) \end{aligned} \tag{8}$$

We formulate the following **lemma:**

Lemma 1: Each of the directed distances between two sets is determined as the greater of the distances between the subsets "of the same names", i.e.

$$h(A,B) = max(h(A_1,B_1),\ h(A_2,B_2)) \tag{9.1}$$

$$h(B,A) = max(h(B_1,A_1),\ h(B_2,A_2)) \tag{9.2}$$

Proof: According to the definition [4] $h(A,B) = \max\limits_{a \in A} \min\limits_{b \in B} \|a-b\|$, but $A = A_1 + A_2$ and $B = B_1 + B_2$, which allows to write:

$$h(A,B) = \max_{a \in A} min\left(\min_{b_1 \in B_1} \|a - b_1\|,\ \min_{b_2 \in B_2} \|a - b_2\| \right) =$$

$$= max \begin{pmatrix} \max\limits_{a_1 \in A_1}\left(min\left(\min\limits_{b_1 \in B_1} \|a_1 - b_1\|,\ \min\limits_{b_2 \in B_2} \|a_2 - b_2\| \right)\right), \\ \max\limits_{a_2 \in A_2}\left(min\left(\min\limits_{b_1 \in B_1} \|a_2 - b_1\|,\ \min\limits_{b_2 \in B_2} \|a_2 - b_2\| \right)\right) \end{pmatrix},$$

but $\min\limits_{b_1 \in B_1} \|a_1 - b_1\| < \min\limits_{b_2 \in B_2} \|a_1 - b_2\|$ and $\min\limits_{b_2 \in B_2} \|a_2 - b_2\| < \min\limits_{b_1 \in B_1} \|a_2 - b_1\|$, according to the way of these subsets forming - in our case, according to the criteria (2.1) and (2.2) or (4.1) and (4.2).

Hence,

$$min\left(\min_{b_1 \in B_1} \|a_1 - b_1\|,\ \min_{b_2 \in B_2} \|a_1 - b_2\| \right) = \min_{b_1 \in B_1} \|a_1 - b_1\| \ .$$

Analogously:

$$min\left(\min_{b_1 \in B_1} \|a_2 - b_1\| \ , \ \min_{b_2 \in B_2} \|a_2 - b_2\| \right) = \min_{b_2 \in B_2} \|a_2 - b_2\| \ .$$

Thus, we can write:

$$h(A,B) = max\left(\max_{a_1 \in A_1} \min_{b_1 \in B_1} \|a_1 - b_1\| \ , \ \max_{a_2 \in A_2} \min_{b_2 \in B_2} \|a_2 - b_2\| \right),$$

but $\max_{a_1 \in A_1} \min_{b_1 \in B_1} \|a_1 - b_1\|$ is nothing else than $h(A_1, B_1)$ and $\max_{a_2 \in A_2} \min_{b_2 \in B_2} \|a_2 - b_2\|$ is nothing else than $h(A_2, B_2)$. Therefore: $h(A,B) = max(h(A_1,B_1), h(A_2,B_2))$, which we want to prove.

Analogously we can prove that

$$h(B,A) = max(\ h(B_1,A_1),\ h(B_2,A_2)).$$

The just proved claims give us a right to write the equation (8) as follows:

$$\begin{aligned} H(A,B) &= H(A_1 + A_2, B_1 + B_2) = \\ &= max\begin{pmatrix} max(\ h(A_1,B_1),\ h(A_2,B_2)), \\ max(h(B_1,A_1),\ h(B_2,A_2)) \end{pmatrix} \end{aligned} \quad (10)$$

The last equation (10) is the base on which we formulate the following **definition:**

Definition 3: Modified Hausdorff distance is defined as:

$$H_M(A,B) = max\begin{pmatrix} max(\ h(A_1,B_1),\ h(A_2,B_2)), \\ max(h(B_1,A_1),\ h(B_2,A_2)) \end{pmatrix} \quad (11)$$

where: $A = A_1 + A_2$; $A_1 = \{a_{11}, a_{12}, ..., a_{1p_1}\}$; $A_2 = \{a_{21}, a_{22}, ..., a_{2p_2}\}$; $p_1 + p_2 = p$ and $p_1, p_2 \neq 0$; $B = B_1 + B_2$; $B_1 = \{b_{11}, b_{12}, ..., b_{1q_1}\}$; $B_2 = \{b_{21}, b_{22}, ..., b_{2q_2}\}$; $q_1 + q_2 = q$; and $q_1, q_2 \neq 0$.

A_1, B_1 and A_2, B_2 satisfy ones and the same criteria.

Properties, which are valid for the classical Hausdorff distance, are valid for the modified distance (11) as well.

1) It is *not negative,* i.e.

$$H_M(A,B) = H_M(A_1 + A_2,\ B_1 + B_2) \geq 0;$$

2) It is *identical*, which means that the distance between two identical sets is zero, i.e.

$$H_M(A,A) = H_M(\ A_1 + A_2, A_1 + A_2\ ) = 0;$$

3) It is *symmetrical*, which means that:

$$\begin{aligned} &H_M(A,B) = H_M(\ B,A\ ) = H_M(A_1 + A_2,\ B_1 + B_2\ ) = \\ &= H_M(B_1 + B_2,\ A_1 + A_2\ ) \end{aligned}$$

Proofs of these properties are trivial and we don't consider them. We pay more attention to the following property, which is used when three sets of points are compared and the distances between them are evaluated. This property is:

4) The *triangle inequality*:

$$H_M(\ A,B\ ) + H_M(\ B,C\ ) \geq H_M(\ A,C\ )$$

or

$$\begin{aligned} &H_M(\ A_1 + A_2, B_1 + B_2\ ) + H_M(\ B_1 + B_2, C_1 + C_2\ ) \geq \\ &\geq H_M(\ A_1 + A_2, C_1 + C_2\ ) \end{aligned} \quad (12)$$

The sense of this inequality is the following: if the distance between the compared sets *A* and *B* is small and if the distance between the sets *B* and *C* is small as well, than it can be claimed that the distance between *A* and *C* is also small, i.e. images, represented by *A* and *C* are similar.

In order to estimate if the triangle inequality (12) is satisfied when the modified Hausdorff distance is used, let answer the question: if the both of the compared subsets are the same, will the similarity measure work properly. With the other words, is there any possibility to make a wrong conclusion, that both sets are similar, i.e. the searched template to be recognized into the image.

In this connection we formulate the following **lemma:**

Lemma 2: If $A = A_1 + A_2$, $B = B_1 + B_2$ and $A_1 \equiv B_1$, but $A_2 \neq B_2$, then

$$H_M(A,B) = max(\ h(A_2,B_2),\ h(B_2,A_2))$$

or if $A_1 \neq B_1$, but $A_2 \equiv B_2$, then

$$H_M(A,B) = max(\ h(A_1,B_1),\ h(B_1,A_1)),$$

where A_1, A_2, B_1, B_2 are not empty sets and they are formed by ones and the same criteria.

Proof: According to Eq. (9.1)

$$h(A,B) = h(A_1 + A_2,\ B_1 + B_2\ ) = max(h(A_1,B_1), h(A_2,B_2)).$$

If $A_1 \equiv B_1$, then $h(A_1,B_1) = 0$, because of identity. If $A_2 \neq B_2$, then $h(A_2,B_2) > 0$, because of no negativity. Hence, $h(A,B) = h(A_2,B_2)$. According to identity and symmetry $h(B_1,A_1) = 0$, and $h(B_2,A_2) > 0$. It follows that we can express Eq. (9.2) as:

$$h(B,A) = h(B_1 + B_2,\ A_1 + A_2\ ) = h(B_2,A_2).$$

Finally, in this case the Hausdorff distance is:

$$H_M(A,B) = H_M(A_1 + A_2, B_1 + B_2) = max\left(h(A_2,B_2),\ h(B_2,A_2) \right).$$

Thus, the first part of the Lemma 2 is proved. The second part can be proved analogously.

The claim from the Lemma 2 can be expressed in the following way as well: the distance between two sets consisting of two subsets, which are obtained by ones and the same criteria, is estimated with the distance between the different subsets "of the same names", if the other subsets "of the same names" coincide.

From this formulation it obviously follows that the triangle inequality is satisfied, since the decision about the closeness between two sets makes only if their different subsets are close enough.

The formulated in this section definitions and lemmas are in the base of our decision for using the Hausdorff distance in a new way, in order to improve its computational complexity.

IV. Computational Complexity of the Modified Hausdorff Distance

According to [4] the computational complexity of the classical Hausdorff distance is $O_H(pq)$, where p and q are the numbers of elements of the compared sets.

Considering Eqs. (7.1) and (7.2) we estimate computational complexity of Hausdorff distance as follows:

$$\begin{aligned} O_H(pq) &= O_H((p_1+p_2).(q_1+q_2)) = \\ &= O_H(p_1q_1+p_1q_2+p_2q_1+p_2q_2) = \\ &= O_H((p_1q_1+p_2q_2)+(p_2q_1+p_1q_2)) = \\ &= O_H(p_1q_1+p_2q_2)+O_H(p_2q_1+p_1q_2) \end{aligned} \tag{13}$$

If the sets $A = A_1 + A_2$ and $B = B_1 + B_2$ satisfy criteria (2.1), (2.2) or (4.1), (4.2), then, according to Lemma 1 it is not necessary to compare the subsets with "different names" (A_1 with B_2, and A_2 with B_1). Thus, we can write:

$$O_{H_M}(p_1q_2+p_2q_1) = 0$$

and

$$O_{H_M}(pq) = O_{H_M}(p_1q_1+p_2q_2) \tag{14}$$

Comparing (13) and (14), it obtains:

$$O_H(pq) = O_{H_M}(p_1q_1+p_2q_2)+O_H(p_2q_1+p_1q_2),$$

from where it follows that:

$$O_{H_M}(pq) = O_H(pq)-O_H(p_2q_1+p_1q_2),$$

i.e.

$$O_{H_M}(pq) < O_H(pq) \tag{15}$$

Inequality (15) shows that the computational complexity of the modified Hausdorff distance is smaller than that of the classical distance, which is our goal.

V. Conclusion

In the present paper we theoretically propose a modification of Hausdorff distance, used for image comparison, considering the specific features of the compared sets of representtative points. We also propose two way for forming these sets, using the before proposed by us methods for representative points extraction [6], [7], [8]. We prove that the computational complexity of the modified Hausdorff distance is smaller than this of the classical distance.

Our next efforts will be directed to developing and investigating different algorithms for evaluation the proposed modified Hausdorff distance and its application in the tasks for searching and localizing the template into a larger image.

References

[1] B. S. Reddy, B. N. Chatterji, An FFT-Based Technique for Translation, Rotation and Scale-Invarian Image Registration, *IEEE Transactions on Image Processing,* Vol. 5, No8, pp. 1266-1271, August 1996

[2] D. M. Gavrila, Multi-feature Hierarhical Template Matching Using Distance Transforms, *in Proc. of IEEE International Conference on Pattern Recognition,* Brisbane, Australia, 1998.

[3] D. M. Gavrila, V. Philomin, Real-Time Object Detection for "Smart" Vehicles, *Proc. of IEEE International Conference on Computer Vision,* pp. 87-93, Kerkyra, 1999.

[4] D. P. Huttenlocher D. P., G. A. Klanderman, W. J. Rucklidge, Comparing Images Using the Hausdorff Distance, *IEEE Transactions on Pattern Analysis and Machine Intelligence,* Vol.15, No9, Sep.1993

[5] D. S. Tyanev, Algorithms for pattern recognition and their application in technical diagnostics, PhD. Thesis, Sofia, 1991

[6] D. S. Tyanev, Y. Petkova, A New Method for Important Points Extraction, *International Scientific Conference CompSysTech 2004,* Rousse, Bulgaria, 17-18 June 2004, pp. III.11-1 – 11-8.

[7] D. S. Tyanev, Y. Petkova. About the Possibilities of a New Edge Definition in Black and White Images, *ISSE 2005, 28th International Spring Seminar on Electronics Technology,* Wiener Neustadt, Austria, May 19-22, 2005.

[8] Y. P. Petkova, D. S. Tyanev, *"An addition to the equipotential method for representative point extraction"*, Journal "Computer Science and Technologies", pp. 44-49, Varna, 2006.