# Analysis of Complex Hadamard Transform Properties 

Rumen P. Mironov ${ }^{1}$, Roumen K. Kountchev ${ }^{2}$


#### Abstract

An analysis of Complex Hadamard Transform properties for 1D and 2D signals is presented. The "basis" images for 2D CHT and energy spectrums are obtained and evaluation of coefficients distribution in complex spectrum space for test images is made.


Keywords - digital signal processing, orthogonal transforms, Hadamard transform, image analysis.

## I. Introduction

Discrete orthogonal transforms [1], [2] have been used extensively in the area of N dimensional signal processing, spectral analysis, pattern recognition and etc. The Walsh Hadamard Transform is a fairly simple transform and has found applications in data compression and watermarking involving image transmission, storage and security. The idea of using complex, rather than integer transforms matrices for spectral processing and analysis has been shown in [3], [4], [5] and [6]. From the Complex Hadamard Transform (CHT), several complex decisions diagrams are derived.

In this paper analysis of more general Complex Hadamard Transform properties for 1D and 2D signals are investigated, which are important for their applications in digital signal processing area. The 'basis' images for 2D CHT are obtained by using a developed MATLAB program simulation. The similar properties with well-known Hadamard Transform are received, what show that CHT can be used by the same way in more complicated analysis and processing [5], [6], [7].

## II. MATHEMATICAL DESCRIPTION

## One dimensional Complex Hadamard Transform

The forward and inverse one dimensional Complex Hadamard Transform (1D CHT) can be represented by the following equations [6]:

$$
\begin{align*}
& y(u)=\sum_{v=0}^{N-1} c(u, v) \cdot x(v)=y_{\mathrm{Re}}(u)+j y_{\mathrm{Im}}(u) \\
& x(v)=\frac{1}{N} \sum_{u=0}^{N-1} c^{*}(u, v) \cdot y(u) \text { for } \quad u, v=\overline{0, N-1} \tag{1}
\end{align*}
$$

where: $N=2^{n} ; y(u), x(v)$ are input and output $N$ dimensional discrete signals; $j=\sqrt{-1} ; y_{\mathrm{Re}}(u), y_{\mathrm{Im}}(u)$ are the real and the imaginary parts of $y(u)$; the coefficients of CHT are:

[^0]\[

\left\lvert\, $$
\begin{align*}
& c(u, v)=j^{u v} s(u, v)  \tag{2}\\
& c^{*}(u, v)=(-j)^{u v} s(u, v)
\end{align*}
$$\right.
\]

$$
s(u, v)=\left\{\begin{array}{lc}
1 & \text { for } n=2  \tag{3}\\
& \sum_{\left.(-1)^{r=3}\left\lfloor u / 2^{r-1}\right\rfloor \nu / 2^{r-1}\right\rfloor} \text { for } n=3,4,5 \ldots . .
\end{array}\right.
$$

is the sign function. Here $L$.$\rfloor is an operator, which represents$ the integer part of the result, obtained after the division.

From the equations (1),(2) and (3) follow that for $u=2 p$

$$
\begin{align*}
y(2 p) & =\sum_{v=0}^{N-1} j^{2 p v} s(2 p, v) x(v)=  \tag{4}\\
& =\sum_{v=0}^{N-1}(-1)^{p v} s(2 p, v) x(v)=y_{\operatorname{Re}}(2 p)
\end{align*}
$$

and in the result $M(2 p)=|y(2 p)|$ and $\varphi(2 p)=0$, where $M($. and $\varphi($.$) are amplitude and phase frequency spectrum$ coefficients respectively.

When $u=4 p+3$ for the odd coefficients of the transform the following are fulfilled:

$$
\begin{align*}
y(4 p+3) & =\sum_{v=0}^{N-1} j^{4 p v} j^{3 v} s(4 p+3, v) x(v)=  \tag{5}\\
= & \sum_{v=0}^{N-1}(-j)^{v} j^{4 p v} s(4 p+3, v) x(v)=y^{*}(4 p+1)
\end{align*}
$$

Equations (4) and (5) are valid for rows $u$ as well for columns $v$ and therefore the coefficients of CHT are symmetric about rows and columns.

In the results can be summarized that the even coefficients of CHT are real and the odd one are complex conjugated.

From the equations (2) the CHT basis matrix of order $2^{\text {n }}$ can be calculated for $\mathrm{n}=2$ :
$\left[\mathrm{CH}_{4}\right]=\left[\begin{array}{cccc}1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j\end{array}\right]\left[\mathrm{CH}_{4}\right]^{*}=\left[\begin{array}{cccc}1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j\end{array}\right]$
The basis matrices of order $2^{n}(\mathrm{n}>2)$ can be received as the Kroneker product of a number of identical "core" matrices of order $2^{n-1}$ in the following way:

$$
\left[\mathrm{CH}_{2^{n}}\right]=\left[\begin{array}{cc}
{\left[\mathrm{CH}_{2^{n-1}}\right]} & {\left[\mathrm{CH}_{2^{n-1}}\right]}  \tag{7}\\
{\left[\mathrm{CH}_{2^{n-1}}\right]} & -\left[\mathrm{CH}_{2^{n-1}}\right]
\end{array}\right] \text {. }
$$

As a sample, the basis Complex Hadamard Transform matrix of order 8 , calculated by the equation (7) is :

$$
\left[C H_{8}\right]=\left[\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & j & -1 & -j & 1 & j & -1 & -j \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -j & -1 & j & 1 & -j & -1 & j \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & j & -1 & -j & -1 & -j & 1 & j \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & -j & -1 & j & -1 & j & 1 & -j
\end{array}\right] .
$$

In conclusion, using (1) forward and inverse CHT can be generalized in matrix form as:

$$
\left\lvert\, \begin{align*}
& \vec{Y}=\left[C H_{N}\right] \vec{X}  \tag{8}\\
& \vec{X}=\frac{1}{N}\left[C H_{N}\right] \vec{Y}
\end{align*}\right. \text { for: }\left\{\begin{array}{l}
\vec{Y}=\{y(u) / u=\overline{0, N-1}\} \\
\vec{X}=\{x(v) / v=\overline{0, N-1}\}
\end{array}\right.
$$

From the above equations the following mathematical properties can be established:

$$
\begin{align*}
& \operatorname{det}\left[\mathrm{CH}_{N}\right]^{2}=N^{N}  \tag{9}\\
& {\left[\mathrm{CH}_{N}\right]\left[\mathrm{CCH}_{N}\right]^{*}=N[I]}  \tag{10}\\
& {\left[C H_{N}\right]^{-1}=\frac{1}{N}\left[C H_{N}\right]^{*}}  \tag{11}\\
& {\left[C H_{N}\right]\left[\left[C H_{N}\right]^{*}=\left[C H_{N}\right]^{t} \cdot\left[C H_{N}\right]=N[I]\right.} \tag{12}
\end{align*}
$$

## Two dimensional Complex Hadamard Transform

The common results, obtained from the one dimensional Complex Hadamard Transform can be generalized for twodimensional Complex Hadamard Transform. In this case the 2 D signals (images) can be represented by the input matrix [ $X$ ] with the size $N \mathrm{x} N$. The result is a spatial spectrum matrix $[Y]$ with the same size. The corresponding equations for the forward and the inverse 2D CHT are:

$$
\left\lvert\, \begin{align*}
& {[Y]=\left[C H_{N}\right][X]\left[C H_{N}\right]} \\
& {[X]=\frac{1}{N^{2}}\left[C H_{N}\right][Y]_{\left[C H_{N}\right]}} \tag{13}
\end{align*}\right.
$$

The symmetry of CHT coefficients allows 2D CHT to be accomplished in two steps. The first one is 1D CHT for every row the image and the second one is 1D CHT for the columns. This difference of transformation makes easier the calculations and the symmetry guarantees that the correlations between image elements in horizontal and vertical direction will influence in the same way the determination of transformed elements. The same considerations can be made for two steps calculation of the inverse 2D CHT.

## III. EXPERIMENTAL RESULTS

The 2D CHT can be expressed in different way by the equation:

$$
\begin{equation*}
[X]=\sum_{k=0}^{N-1} \sum_{l=0}^{N-1} y_{k l}\left[T_{k l}\right] \tag{14}
\end{equation*}
$$

where $\left[T_{k l}\right]$ is the matrix of "basis" image with consecutive number ( $k, l$ ). This expression can be presented as image decomposition $[X]$ in order on $N^{2}$ "basis" images with weighted coefficients $y_{k l}$. Using equations (6) and (14) these images of order 4 are simulated on MATLAB and are shown in Fig. 1a.


Fig. 1a. "Basis" images of order N=4 for 2D CHT.
In this figure the values $(+1)$ are colored with white level, the values $(-1)$ - with black level, the values $(+\mathrm{j})$ - with white gray level and the values ( -j ) - with dark gray level. It is show that the received "basis" images look like the non-ordered "basis" images of real Hadamard Transform (HT).


Fig.1b. "Basis" images of order $\mathrm{N}=4$ for 2D HT.
On Fig. 1b are shown the same "basis" images simulated on MATLAB by real Hadamard Transform, arranged in naturally order.

On Fig. 2 are shown the "basis" images for 2D CHT of order 8 , received in the same way.


Fig.2. Basis images of order 8 for 2D CHT.

For the analyses of spectral distribution between the coefficients of 2D CHT a test image "LENNA", shown in Fig.3, with size $512 \times 512$ and 256 gray levels is used. This image is transformed by the 2D CHT with kernel $16 \times 16$. By this way the input image is divided on 1024 sub-images with size $16 \times 16$ and is calculated by MATLAB 6.5 program. In Fig.4a and Fig. 4b the averaged amplitude frequency spectrums of all sub-blocks for two-dimensional Complex Hadamard Transform and two dimensional real Hadamard Transform respectively, are shown. On Fig. 4c the averaged phase frequency spectrum calculated for all sub-blocks, for twodimensional Complex Hadamard Transform, is shown.

## IV. Conclusion

A class of Complex Hadamard Transformation is presented. The general principles of complex matrices construction of high order for 1D and 2D transforms are given. The basic properties of CHT are discussed. The obtained amplitude spectrums for CHT and HT are practically identical and show that both can be used in similar applications.

The presented Complex Hadamard Transform can be used in digital signal processing for spectral analysis, pattern recognition, digital watermarking, coding and transmission of one-dimensional and two-dimensional signals.

## ACKNOWLEDGEMENT

The authors thank the National Fund for Scientific Research of the Bulgarian Ministry of Education and Science for the financial support by the contract I-1304/2003.

## REFERENCES

[1] N. Ahmed, K. R. Rao. Orthogonal Transforms for Digital Signal Processing, Springer-Verlag Berlin, Heidelberg, 1975.
[2] W. K. Pratt. Digital Image Processing, John Wiley\&Sons, N.Y., 2001.
[3] B. Falkowski, S. Rahardja. "Complex Spectral Decision Diagrams", Proc. of the $26^{\text {th }}$ Int. Symposium on Multiple Valued Logic, Vol. ISMVL’96, 1996.
[4] S. Rahardja, B. Falkowski. "Complex Composite Spectra of Unified Complex Hadamard Transform for Logic Functions", IEEE Trans. on Circuits and Systems-II: Analog and Digital Signal Processing, Vol. 47, No. 11, November 2000.
[5] B. Falkowski, S. Rahardja. "Complex Hadamard Transforms: Properties, Relations and Architecture", IEICE Trans. Fundamentals. Vol. E87-A, No.8, August 2004.
[6] R. Kountchev, R. Mironov. "Audio Watermarking in the Phase-Frequency Domain", XL Intern. Scientific Conference on Information, Communication and Energy Systems and Technologies, ICEST'2005, Nis, Serbia and Montenegro, 2005.
[7] A. D. Poularikas. The Transforms and Applications Handbook, Second Ed., CRC Press, 2000.


Fig. 3. Test image "LENNA" with size $512 \times 512$ pixels and 256 gray levels.


Fig. 4. Averaged amplitude and phase spectrums of image "LENNA"


[^0]:    ${ }^{1}$ Rumen P. Mironov is with the Faculty of Communication Technics and Technology, Technical University of Sofia, Kl. Ohridsky 8, 1000 Sofia, Bulgaria, E-mail: rpm@tu-sofia.bg
    ${ }^{2}$ Roumen K. Kountchev is with the Faculty of Telecommunications, Technical University of Sofia, Boul. Kl. Ohridsky, 8, Sofia 1000, Bulgaria. Email: rkountch@tu-sofia.bg

