Optimal Product Pyramid Vector Quantization of Memoryless Laplacian Source

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Abstract – **In this paper the optimal product pyramid Z^{n-1} lattice vector quantization of memoryless Laplacian source will be considered. The asymptotic analysis based on minimum distortion criterion will be performed. We will show that obtained optimal distortion is function of scalar case Bennet's integral and on this base we will find optimal multidimensional radial companding characteristic. In that way, applying the quantization technique that has small implementation complexity, we will achieve the signal to quantization noise ratio which in some cases differs from that of optimal vector quantization for about 0.5 dB.**

Keywords – **Optimal product vector quantization, Z^{n-1} lattice based quantization, optimal multidimensional radial companyding characteristic**

I. INTRODUCTION

In order to achieve the conversion of analog signals into efficient digital representation and the compression of digital information into the fewest possible bits, the vector quantization was developed. Vector quantization is an extension of the simple scalar quantization to multidimensional spaces and it can yield smaller average mean squared error per dimension than scalar quantization for the case of fine quantization [1], [2]. Also, the rapid advance in digital signal processor chips made possible low cost implementation of complex vector coding techniques.

The quantizer presented in this paper is designed for a memoryless Laplacian source. One reason for studying the memoryless Laplacian source is that it naturally arises in numerous applications. For example, the first approximation to the long-time-averaged probability density function (pdf) of speech amplitudes is provided by Laplacian model [1]. Also, in a number of papers the vector quantization of memoryless Laplacian source was analyzed since the probability density function of the difference signal for an image waveform follows the Laplacian function [3].

In this paper we consider the optimal product vector quantization which in comparisson with the unrestricted optimal vector quantization has a little worse performances but the lower implementation complexity. One quantization technique that can be applied for quantizer design is suggested in [4]. In that paper the importance of source geometry and lattice quantization was noted. Author considered the weighted pyramid vector quantizers for Laplacian sources and used one established approximate heuristic distortion formula for optimal weighted pyramid vector quantization. Here we derive the exact equation for the optimal total distortion.

Multidimensional companding on optimal high rate quantization was introduced by Gersho [5]. He pointed at the difficulties of doing optimal quantization with companding vector quantization. Bucklew also showed that asymptotically optimal unrestricted vector quantizers for vector dimensions 3 and greater can not be implemented using a companding structure, except for a very restricted class of source densities [6], [7]. A significant contribution to the understanding of companding was made in [8]. In [8] and [9] one analytic solution for radial companding characteristic was suggested. The proposed solution is independent of source type. However, author of [10] showed that in the case of Gaussian source quantization, the signal to quantization noise ratio obtained applying the companding characteristic obtained in [8] and [9] is for about 2.5 dB smaller than that of optimal vector quantization and he gave the better solution. In [10] the proposed heuristic solution gives performance for about 1 dB worse than that of optimal vector quantization in the case of Gaussian source quantization.

In order to respect source geometry we observe the pyramid vector quantization. Also, since we observe the equal points number per hyperpyramid equally distributed, all these points would map into the same points on hyperpyramid that has radius 1. Because of that it is convenient to apply the Z^{n-1} lattice on unit pyramid surface and after that radial map the obtained n-1 dimensional cells into n dimensional cells. Then we perform the asymptotic analysis based on minimum distortion criterion. After some mathematical manipulation we express the normalized moment of n dimensional cell in function of the normalized moment of n-1 dimensional cell projection. Than, by optimizing the total distortion, we find the optimal hyperpyramid number and distortion. We show that obtained distortion is function of scalar case Bennet's integral and on this base we find the optimal multidimensional radial companding characteristic. In that way, applying the quantization technique that has smaller implementation complexity, we achieve the signal to quantization noise ratio which in some cases differs from that of optimal vector quantization for about 0.5 dB.

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II. Asymptotic Analysis of Optimal Product Pyramid Vector Quantization

For n-dimensional vector $\mathbf{x} = [x_1 \quad x_2 \quad \cdots \quad x_n]^T$ consisting of independent and identically distributed (i.i.d.) Laplacian variables x_i with zero mean and unit variance, the joint probability density function of $\boldsymbol{x}$ is

$$f_{\mathbf{x}}(\mathbf{x}) = \prod_{i=1}^{n} f(x_i) = (2)^{-\frac{n}{2}} \cdot \exp\left(-\sqrt{2}\sum_{i=1}^{n}|x_i|\right). \tag{1}$$

The contour of constant probability density function (pdf) is given by

$$\sum_{i=1}^{n}|x_i| = -\frac{\sqrt{2}}{2}\ln\left[f_c\, 2^{\frac{n}{2}}\right] \equiv g_0,\ g_0 \geq 0, \tag{2}$$

where f_c is the value of pdf. This is an expression for the n-dimensional pyramid with radius g_0, where we define the radius as

$$g = \sum_{i=1}^{n}|x_i| = \|\mathbf{x}\|_n^1. \tag{3}$$

The radius g is also random variable and has a pdf given as

$$f_g(g) = \frac{2^{\frac{n}{2}}}{\Gamma(n)} g^{n-1} e^{-\sqrt{2}g},\quad g \geq 0. \tag{4}$$

Using the previous formulation for g, we can perform the coordinate transformation and express the vector $\boldsymbol{x}$ through the vector intensity, i.e. the vector amplitude g and the location vector $\mathbf{s} = [s_1 \quad s_2 \quad \cdots \quad s_n]^T$

$$\mathbf{x} = g\mathbf{s}. \tag{5}$$

The n dimensional vector $\boldsymbol{s}$ is projection of vector $\boldsymbol{x}$ on hyperpyramid that has radius 1, i.e. one endpoint of vector $\boldsymbol{s}$ is placed on unit hyperpyramid.

In order to respect source geometry, we consider that a nonlinear vector quantization of Laplacian source has L representative amplitude levels determined with equation $\hat{g}_i = const.,\quad i = 1,...,L$. In accordance with this assumption, the decision amplitude levels are g_i, $i = 0,\ldots, L$. Also, since we analyze the product vector quantization, we have the equal points number per amplitude levels M equally distributed. Therefore, we have that

$$LM = N, \tag{6}$$

where N is total points number of quantizer. If we take into consideration that number of amplitude levels L is very large, the required memory in the case of product nonuniform quantization is far less than that in the case of unrestricted nonuniform quantization when we should store the points number on each amplitude levels M_i, $i = 1,\ldots,L$.

In the case of product quantization the points from different amplitude levels map into same points on hyperpyramid that has radius 1. Because of that we can apply the Z^{n-1} lattice on unit pyramid surface and after that radial map the obtained n-1 dimensional cells into n dimensional cells with representatives equally distributed on observed hyperpyramides (Fig 1.). So, if we sign the quantization cell as $c_{i,j}$, where i determines amplitude level, i=1,…, L, j the cell on given representative amplitude level, j=1,…, M, we can write that representative of cell $c_{i,j}$ is

$$\hat{\mathbf{x}}_{i,j} = \hat{g}_i \hat{\mathbf{s}}_j,\quad j = 1,...,M,\ i = 1,...,L. \tag{7}$$

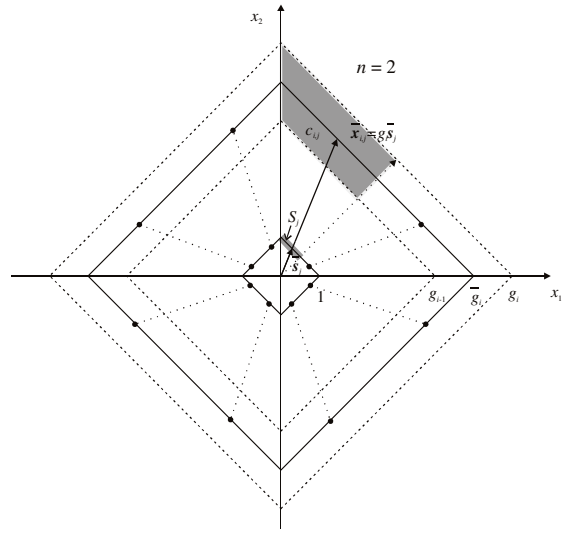

Fig. 1. Illustration of pyramid Z^{n-1} lattice vector quantization.

Now, we should define the normalized second order moment, i.e. the inertia moment of cell $c_{i,j}$

$$M^n = \frac{1}{n}\frac{1}{\mathrm{vol}(c_{i,j})^{1+\frac{2}{n}}}\int_{c_{i,j}} d(\mathbf{x}, \hat{\mathbf{x}}_{i,j})d\mathbf{x}, \tag{8}$$

where vol($c_{i,j}$) is the volume of the cell $c_{i,j}$

$$\mathrm{vol}(c_{i,j}) = \frac{2^n}{M\Gamma(n+1)}\left(g_i^n - g_{i-1}^n\right) \tag{9}$$

and $$d(\mathbf{x}, \hat{\mathbf{x}}_{i,j}) = \left(\|\mathbf{x}\|_n^1\right)^2 + \left(\left\|\hat{\mathbf{x}}_{i,j}\right\|_n^1\right)^2 - 2\|\mathbf{x}\|_n^1\left\|\hat{\mathbf{x}}_{i,j}\right\|_n^1 \cos\varphi. \tag{10}$$

If we take into consideration that corresponding unit vectors to vectors $\boldsymbol{x}$ and $\hat{\mathbf{x}}_{i,j}$ are $\boldsymbol{s}$ and $\hat{\mathbf{s}}_j$, respectively, the Eq. (10) also applies to them

$$d(\mathbf{s}, \hat{\mathbf{s}}_j) = 2 - 2\cos\varphi. \tag{11}$$

Beside this, for great number of amplitude levels L, we can assume that $g \approx \hat{g}_i$ inside one cell, i.e. we can write

$$d(\mathbf{x}, \hat{\mathbf{x}}_{i,j}) = \left(g - \hat{g}_i\right)^2 + \hat{g}_i^2 d(\mathbf{s}, \hat{\mathbf{s}}_j). \tag{12}$$

We also know that for great number of amplitude levels L, the volume of cell $c_{i,j}$ can be approximated as

$$\mathrm{vol}(c_{i,j}) \approx \frac{2^n}{M\Gamma(n)}\hat{g}_i^{n-1}\Delta g_i = S(\hat{g}_i)\Delta g_i, \tag{13}$$

where $S(\hat{g}_i)$ is effective surface of one cell from hyperpyramid that has radius $\hat{g}_i$ and $\Delta g_i = g_i - g_{i-1}$.

Similar, we can define the normalized second order moment of $c_{i,j}$ cell projection on unit hyperpyramid S_j

$$G(\Lambda)=\frac{1}{n-1}\frac{1}{\mathrm{vol}(S_j)^{1+\frac{2}{n-1}}}\int_{S_j} d(\mathbf{s},\hat{\mathbf{s}}_j)ds\,, \tag{14}$$

where n-1 is dimension of cell S_j and

$$\mathrm{vol}(S_j)=S(\hat{g}_i=1)=\frac{2^n}{M\Gamma(n)}\,. \tag{15}$$

If we substitute Eq. (12) in Eq. (8) simultaneously using the relation $d\mathbf{x}=\hat{g}_i^{n-1}dgd\mathbf{s}$, after some mathematical manipulation we express the normalized moment of n dimensional cell in function of the normalized moment of n-1 dimensional cell projection

$$M^n=\frac{1}{n}\frac{1}{\mathrm{vol}(c_{i,j})^{1+\frac{2}{n}}}\cdot \cdot\left[\frac{\Delta g_i^2}{12}+(n-1)G(\Lambda)\left[S(\hat{g}_i=1)\right]^{\frac{2}{n-1}}\hat{g}_i^2\right]. \tag{16}$$

Now, we can find distortion per dimension of the jth cell on the ith representative amplitude level ($c_{i,j}$)

$$D_{i,j}=\frac{1}{n}\int_{c_{i,j}} d(\mathbf{x},\hat{\mathbf{x}}_{i,j})f_{\mathbf{x}}(\mathbf{x})d\mathbf{x},\ j=1,2,...,M_i,\ i=1,2,...,L\,. \tag{17}$$

We perform asymptotic analysis, i.e. we assume that pdf of input vector $f_{\boldsymbol{x}}(\boldsymbol{x})$ is constant inside cell and has value $f_{\mathbf{x}}(\hat{\mathbf{x}}_{i,j})$. Under this assumption we can write that

$$D_{i,j}=M^n\mathrm{vol}(c_{i,j})^{1+\frac{2}{n}}f_{\mathbf{x}}(\hat{\mathbf{x}}_{i,j})=M^n\mathrm{vol}(c_{i,j})^{\frac{2}{n}}P_{i,j}\,, \tag{18}$$

where $P_{i,j}$ is the probability that the vector $\boldsymbol{x}$ belongs to cell $c_{i,j}$

$$P_{i,j}=\int_{c_{i,j}} f_{\mathbf{x}}(\mathbf{x})d\mathbf{x}=f(\hat{\mathbf{x}}_{i,j})\int_{c_{i,j}} d\mathbf{x}=f(\hat{\mathbf{x}}_{i,j})\mathrm{vol}(c_{i,j})\,. \tag{19}$$

After that the granular distortion of cells that have representatives on the ith amplitude representative level is

$$D_i=\sum_{j=1}^{M}D_{i,j}=M^n\mathrm{vol}(c_{i,j})^{\frac{2}{n}}P_i\,, \tag{20}$$

where P_i is a probability that vector $\boldsymbol{x}$ is located between hyperpyramides that have radii g_i and g_{i-1}

$$P_i=\sum_{j=1}^{M}P_{i,j}=\int_{g_{i-1}}^{g_i}f_g(g)dg\approx f_g(\hat{g}_i)\Delta g_i\,. \tag{21}$$

On the other hand, we also observe the nonuniform quantization through the companding vector quantization. Because of that, we define the multidimensional radial companding characteristic $h(g)$. This function maps compressor input range $[0,+\infty)$ into output range $[0,1)$. Since we consider the great number of amplitude levels L, we use following relation in our analysis

$$\left.\frac{dh(g)}{dg}\right|_{g=\hat{g}_i}=\frac{\frac{1}{L}}{\Delta g_i}\,, \tag{22}$$

i.e.

$$\Delta g_i=\frac{1}{L\left.\frac{dh(g)}{dg}\right|_{g=\hat{g}_i}}\,. \tag{23}$$

Substituting the Eqs. (16), (23) and (21) into Eq. (20), we get that distortion per dimension for whole product nonuniform vector quantizer of signal generated by Laplacian source can be written as follows

$$D=\sum_{i=1}^{L}D_i=\frac{1}{L^2 12n}\sum_{i=1}^{L}\frac{f_g(\hat{g}_i)}{\left(\left.\frac{dh(g)}{dg}\right|_{g=\hat{g}_i}\right)^2}\Delta g_i+ +\frac{(n-1)G(\Lambda)\left[S(\hat{g}_i=1)\right]^{\frac{2}{n-1}}}{n}\sum_{i=1}^{L}\hat{g}_i^2 f_g(\hat{g}_i)\Delta g_i\,. \tag{24}$$

If we apply the Riemann integral definition on the right hand of (24), we can replace sum with integral

$$D=\frac{1}{L^2 12n}\int_0^{+\infty}\frac{f_g(g)}{\left(\frac{dh(g)}{dg}\right)^2}dg+ +\frac{(n-1)G(\Lambda)\left[S(\hat{g}_i=1)\right]^{\frac{2}{n-1}}}{n}\int_0^{+\infty}g^2 f_g(g)dg\,. \tag{25}$$

If we substitute Eq. (15) in Eq. (25) and take into consideration that $M = N/L$, we get the expression for total distortion per dimension of product pyramid Z^{n-1} lattice vector quantization D in function of hyperpyramides number L

$$D=\frac{l_0}{L^2}+\frac{tL^{\frac{2}{n-1}}}{N^{\frac{2}{n-1}}}\,, \tag{26}$$

$$l_0=\frac{1}{12n}\int_0^{+\infty}\frac{f_g(g)}{\left(\frac{dh(g)}{dg}\right)^2}dg\,, \tag{27}$$

$$t=\frac{2^{\frac{n+1}{n-1}}(n-1)(n+1)G(\Lambda)}{\left[\Gamma(n)\right]^{\frac{2}{n-1}}}\int_0^{+\infty}g^2 f_g(g)dg\,. \tag{28}$$

Now we can perform the asymptotic analysis based on minimum distortion criterion. By optimizing the total distortion, i.e differentiating D with respect to L and equalizing with zero, we find the optimal hyperpyramid number and distortion

$$L_{opt}=(n-1)^{\frac{n-1}{2n}}l_0^{\frac{n-1}{2n}}t^{-\frac{n-1}{2n}}N^{\frac{1}{n}}\,, \tag{29}$$

$$D_{opt}=n(n-1)^{-\frac{n-1}{n}}l_0^{\frac{1}{n}}t^{\frac{n-1}{n}}N^{-\frac{2}{n}}\,, \tag{30}$$

respectively. Since n, t and N are given parameters, D_{opt} depends only of scalar variable l_0. If we carefully observe the expression for l_0, we see that Eq. (27) corresponds to Bennet's integral form in the case of scalar quantization. Since in this way we show that obtained distortion is only function of scalar case Bennet's integral, we can find optimal multidimensional radial companding characteristic. Namely, we conclude that multidimensional radial companding characteristic has the same form as the optimal scalar companding characteristic

$$h(g)=\frac{\int_0^g x^{\frac{n-1}{3}}\exp\left(-\frac{\sqrt{2}x}{3}\right)dx}{\left(\frac{3}{\sqrt{2}}\right)^{\frac{n+2}{3}}\Gamma\left(\frac{n+2}{3}\right)}\,. \tag{31}$$

In [11] the optimal piecewise uniform vector quantization of memoryless Laplacian source was analyzed. Applying this analysis, the optimal multidimensional radial companding characteristic (31) can be also obtained, but the procedure would be more complicate.

After short mathematical manipulation, we obtain that optimal product Z^{n-1} lattice vector quantization distortion and optimal number of hyperpyramid are

$$D_{opt} = 2^{\frac{n-2}{n}} 3^{\frac{n+1}{n}} n^{\frac{n-1}{n}} (n+1)^{\frac{n-1}{n}} \left[\frac{\Gamma\left(\frac{n+2}{3}\right)}{\Gamma(n)} \right]^{\frac{3}{n}} G(\Lambda)^{\frac{n-1}{n}} N^{-\frac{2}{n}}, \quad (32)$$

$$L_{opt} = 2^{-\frac{2n-1}{n}} 3^{\frac{n^2-1}{2n}} n^{-\frac{n-1}{2n}} (n+1)^{-\frac{n-1}{2n}} \cdot \frac{\Gamma\left(\frac{n+2}{3}\right)^{\frac{3(n-1)}{2n}}}{\Gamma(n)^{\frac{n-3}{2n}}} G(\Lambda)^{-\frac{n-1}{2n}} N^{\frac{1}{n}}. \quad (33)$$

respectively.

In Table I we show the obtained results for signal to quantization noise ratio *SQNR*

$$SQNR = 10 \log \frac{1}{D} \quad (34)$$

in function of dimension n for bit rate

$$R = \frac{1}{n} \log_2 N = 8 \text{ bits/dimension}. \quad (35)$$

Simultaneous the experimental results are obtained ($SQNR^e$). Since the confidence of simulation is 95%, the performed simulation run of 1000 vectors shows good matching with theoretical results.

Table also contains the values of signal to quantization noise ratio in the case of optimal unrestricted nonuniform vector quantization

$$SQNR^{\mathrm{OVQ}} = 10 \log \frac{1}{\frac{1}{6}\left(\frac{n+2}{n}\right)^{n+2} N^{-\frac{2}{n}}}, \quad (36)$$

and difference $\Delta SQNR = SQNR^{\mathrm{OVQ}} - SQNR$. It is obvious from Table I that difference between signal to quantization noise ratio of optimal and optimal product vector quantization decreases with the increase of the dimension n and for $n = 32$, $\Delta SQNR = 0.55$ dB. Furthermore, for large dimension ($n = 125$), optimal product pyramid vector quantization performances converge to those of optimal vector quantization ($\Delta SQNR = 0.19$ dB). On the other hand, comparing Eq. (32) with Eq. (36), we note that $\Delta SQNR$ is independent of N, i.e. bit rate R.

TABLE I
SIGNAL TO QUANTIZATION NOISE RATIO IN FUNCTION OF DIMENSION

N	16	24	32	125
SQNR [dB]	45.82	46.22	46.44	47.00
$SQNR^e$ [dB]	46.54	46.57	46.82	47.11
$SQNR^{\mathrm{OVQ}}$ [dB]	46.74	46.91	46.99	47.19
$\Delta SQNR$ [dB]	0.92	0.69	0.55	0.19
G [dB]	4.12	4.52	4.74	5.3

If we take into consideration that signal to noise ratio of optimal scalar vector quantization is 41.7 dB for 8 bits per sample, we see that obtained gain with our quantizer $G = SQNR - 41.7$ dB is large (the last row of Table I).

III. CONCLUSION

In this paper we perform an exact and complete asymptotic analysis of optimal product pyramid Z^{n-1} lattice vector quantization of memoryless Laplacian source. We derive the expressions for the optimal hyperpyramid number, the optimal total distortion and the optimal multidimensional companding characteristic. The presented analysis is very simple and convenient for implementation. Opposite to classical product pyramid lattice vector quantization the hyperpyramides during this quantization are not equidistant which enables us determination optimal multidimensional radial companding characteristic.

Results show that suggested quantization model gives the signal to quantization noise ratio which in some cases differs from that of optimal vector quantization for about 0.5 dB. It proves that optimal product pyramid Z^{n-1} lattice vector quantization in comparisson with the unrestricted optimal vector quantization has a little worse performances but the lower implementation complexity. In order to simplify implementation of considered vector quantization, the model should be linearized applying the piecewise uniform product vector quantizer.

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