New Experiences in Simple Numerical Modelling of Sommerfeld's Integral Kernel

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***Abstract* – A number of simple expressions used for modelling the Sommerfeld's integral kernel (SIK), while evaluating the EM filed structure in the surroundings of the vertical Hertz's dipole (VHD) placed above a lossy half-space, are presented in this paper. The results obtained applying the approximate SIK models are compared to exact calculations, confirming a very good accuracy of proposed models.**

***Keywords* – Sommerfeld's integral kernel, Vertical Hertz's dipole, Lossy half-space.**

I. INTRODUCTION

In the process of the EM field characterization in the surroundings of vertical wire structures placed above a lossy half-space, a specific class of integrals, in literature referred to as integrals of Sommerfeld's type, describes the influence of the finite ground conductivity. Since these integrals can not be solved in a closed form, two approaches are possible. One considers a direct numerical integration of the Sommerfeld's integrals, which is a very time-consuming and complex way to deal with this type of integrals. The other one considers modelling of the SIK in a certain way in order to obtain an approximate closed form solution, but of satisfying accuracy. Throughout the last century many authors have dealt with this problem and, as a result, great number of expressions was proposed for modelling the SIK.

Authors of this paper have also dealt with simple SIK modelling ([1]-[9]):

- The first group of the SIK models was obtained developing the spectral reflection coefficient (SRC) into geometry progression series and adopting only the first two or three terms of the obtained series. Model derivation was done under an assumption that the refraction coefficient is $n_{10} >> 1$. Fortunately, in the majority of cases of practical interest, the refraction index is $n_{10} > 3$, which is the range where the model was proven to be satisfyingly accurate ([1]-[3]);
- Developing the SRC into Taylor's series around certain characteristic values of the spectral variable, a number of models was also formed. SIK models were obtained adopting the first few terms, which have extended the application range to the cases of smaller values of the refraction coefficient, i.e. $n_{10} > 2$. The accuracy of this model depends on the number of series' terms that are adopted, the value around which the development is done, as well as the type of the considered wire structure and its location ([4]);
- In the third and forth group of new models, the influence of the lossy ground on the EM field structure is described by one or more fictive images. The position of the images and their weight current coefficients are determined in the process of forming the proposed models ([5]-[8]). The forth group of models is a new form of the so-called two-image approximation of the SIK ([9]).

Adopting good sides of described models, the authors of this paper have proposed a number of SIK models [5-9]. Besides their simple mathematical form, these models are characterized as general since they are not limited by the electrical parameters of the lossy half-space. Also, the application of such approximate expressions gives satisfyingly accurate results for the SIK in the near field, which can be verified comparing them to exact calculations from [10] and [11].

II. THEORETICAL BACKGROUND

A. Description of the model geometry

The VHD is placed in the air ($\sigma_0 = 0, \varepsilon_0, \mu_0$) at height z'_k above a linear, isotropic and homogenous lossy half-space of electrical parameters σ_1, $\varepsilon_1 = \varepsilon_{r1}\varepsilon_0$, $\mu_1 = \mu_0$ ($\underline{\sigma}_i = \sigma_i + j\omega\varepsilon_i$, $i = 0,1$ - complex conductivity, $\underline{\gamma}_i = \alpha_i + j\beta_i = (j\omega\mu_i\underline{\sigma}_i)^{1/2}$, $i = 0,1$ - complex propagation constant, $\underline{\varepsilon}_{r1} = \underline{n}_{10}^2 = \varepsilon_{r1} - j\varepsilon_{i1}$ - complex relative permittivity, $\underline{n}_{10}$ - complex refraction index, λ_0 - wavelength in air, and $\omega = 2\pi f$ - angular frequency). Schematic illustration of described VHD is given in Fig.1.

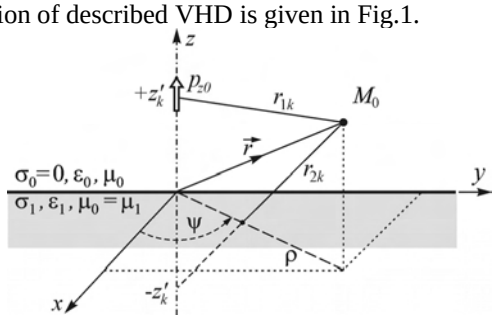

Fig. 1. Schematic illustration of the VHD above a lossy half-space.

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In the described case of the VHD, the SIK that describes the field scattered from the lossy half-space is defined as

$$S_{00}^{v}(r_{2k}) = \int_{\alpha=0}^{\infty} \tilde{R}_{z10}(\alpha)\tilde{K}_0(\alpha, r_{2k})\,\mathrm{d}\alpha\,, \qquad (1)$$

where: $\tilde{R}_{z10}(\alpha) = \tilde{R}_{z10}(u_0) = (\underline{n}_{10}^2 u_0 - u_1)/(\underline{n}_{10}^2 u_0 + u_1)$ - SRC, $\tilde{K}_0(\alpha, r_{2k})$ - spectral form of the standard potential kernel from the image placed at distance $r_{2k} = \sqrt{\rho^2 + (z + z_k')^2}$, and $u_i = (\alpha^2 + \underline{\gamma}_i^2)^{1/2}$, $i = 0,1$.

B. SIK Approximations

The first model assumes the SRC in a form of a rational function with two unknown constants that are determined by matching the SCR at characteristic points:

$$\tilde{R}_{z10}(u_0) \cong B + A\,\frac{\underline{\gamma}_0}{u_0}\,. \qquad (2)$$

Constants A and B are determined in the process of approximation as in [5]. Matching the Eq.(2) at points $u_0 = \underline{\gamma}_0$ and $u_0 \to \infty$, constants A and B are obtained as

$$A = (R_0 - R_\infty) \text{ and } B = R_\infty\,, \qquad (3)$$

where $R_0 = (\underline{n}_{10} - 1)/(\underline{n}_{10} + 1)$, $R_\infty = (\underline{n}_{10}^2 - 1)/(\underline{n}_{10}^2 + 1)$.

Substituting (2) in (1), and adopting Eq. (3), we have:

$$S_{00}^{v}(r_{2k}) \cong BK_0(r_{2k}) + A\underline{\gamma}_0 L(r_{2k})\,, \qquad (4)$$

where $L(r_{2k})$ is a new integral kernel,

$$L(r_{2k}) = \int_{v=z+z_k'}^{\infty} K_0(r_{2kv})\,\mathrm{d}v =$$

$$= -\int_{v=0}^{z+z_k'} K_0(r_{2kv})\,\mathrm{d}v - \frac{\pi}{2}[N_0(\beta_0\rho) + j\,J_0(\beta_0\rho)]\,. \qquad (5)$$

In Eq.(5), $N_0(\beta_0\rho)$ and $J_0(\beta_0\rho)$ are zero-th Neumann and Bessel functions of the first kind, respectively, $\rho = [(x - x_k')^2 + (y - y_k')^2]^{1/2}$ is radial distance, and $r_{2kv} = (\rho^2 + v^2)^{1/2}$.

Comment: Constant A can be obtained matching Eq.(2) at a different point ([6]), for example $u_0 = u_B = \underline{\gamma}_0/(\underline{n}_{10}^2 + 1)^{1/2}$. This results in

$$A = -B/(\underline{n}_{10}^2 + 1)^{1/2}\,. \qquad (6)$$

If different values are adopted for constants A and B, e.g. $A = -2/\underline{n}_{10}$ and $B = 1$, a simple SIK model is also obtained, which actually presents only first two terms of SRC development into a geometry progression series under a condition $n_{10} >> 1$ ([1]-[3]).

The second model is in an exponential form

$$\tilde{R}_{z10}(u_0) \cong B + A\mathrm{e}^{-(u_0 - \underline{\gamma}_0)d}\,. \qquad (7)$$

Again, A and B are unknown constants, and d unknown distance, which are all determined in the process of approximation ([9]). Matching the Eq.(6) at $u_0 = \underline{\gamma}_0$ and $u_0 \to \infty$, and its first derivative at $u_0 = \underline{\gamma}_0$, the following is obtained:

$$A = (R_0 - R_\infty)\,,\ B = R_\infty \text{ and } d = \left|(2/\underline{\gamma}_1)R_0 \big/ (R_0 - R_\infty)\right|. \qquad (8)$$

Substituting (7) in (1), and adopting Eq. (8), the SIK gains the following two-image approximation form:

$$S_{00}^{v}(r_{2k}) \cong BK_0(r_{2k}) + Ae^{jD}K_0(r_{2kd})\,, \qquad (9)$$

where: $r_{2kd} = \sqrt{\rho^2 + (z + z_k' + d)^2}$ - real distance of the second image in the flat mirror to the observed field point M_0, and $D = \beta_0 d$ - electrical length of the distance d.

III. Numerical Results

A number of numerical experiments were performed based on the presented theoretical analysis with an aim to prove the validity of the proposed models. Obtained results are compared to the accurate ones from [10] and [11] and will be graphically illustrated further in the paper. Relative error of accurate results is estimated to be $10^{-7} - 10^{-6}$.

Based on the approximate SIK model given by Eq.(4), modulus of the normalized SIK, S_{00}^{v}/β_0, was determined for different values of the parameters on which it depends, and the results are given in Figs.2a, b and c. Position of the VHD z_k', is taken as a parameter, relative permittivity is $\varepsilon_{r1} = 10$, and Figs. 2a, b and c correspond to the following values of the normalized conductivity $\sigma_1\lambda_0 = 10^{-4}\,S$, $\sigma_1\lambda_0 = 0.175\,S$ and $\sigma_1\lambda_0 = 10\,S$, respectively. In order to investigate accuracy of the proposed model, accurate results from [10] and [11] are also shown (solid circle).

Results obtained applying the proposed model for the SIK are in very good accordance with the ones of accurate calculations, regardless of the electrical parameters of the lossy half-space or the VHD position. In that sense, the proposed model can be, besides simple, characterized as general and accurate.

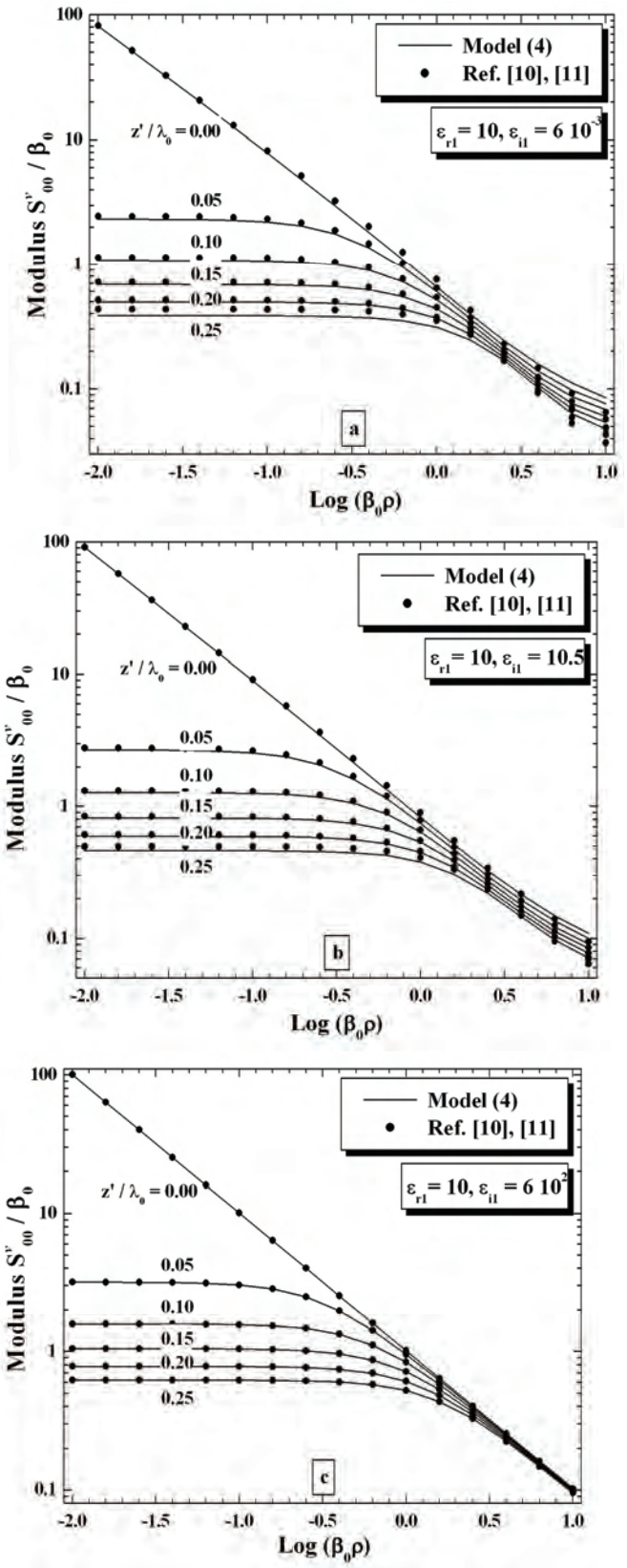

Fig. 2. Modulus of the normalized SIK, at plane $z = 0$, versus normalized radial distance. Relative dielctric constant is $\varepsilon_{r1} = 10$, and VHD position z'_k and imaginary part of dielectric permittivity ε_{i1} are parameters.

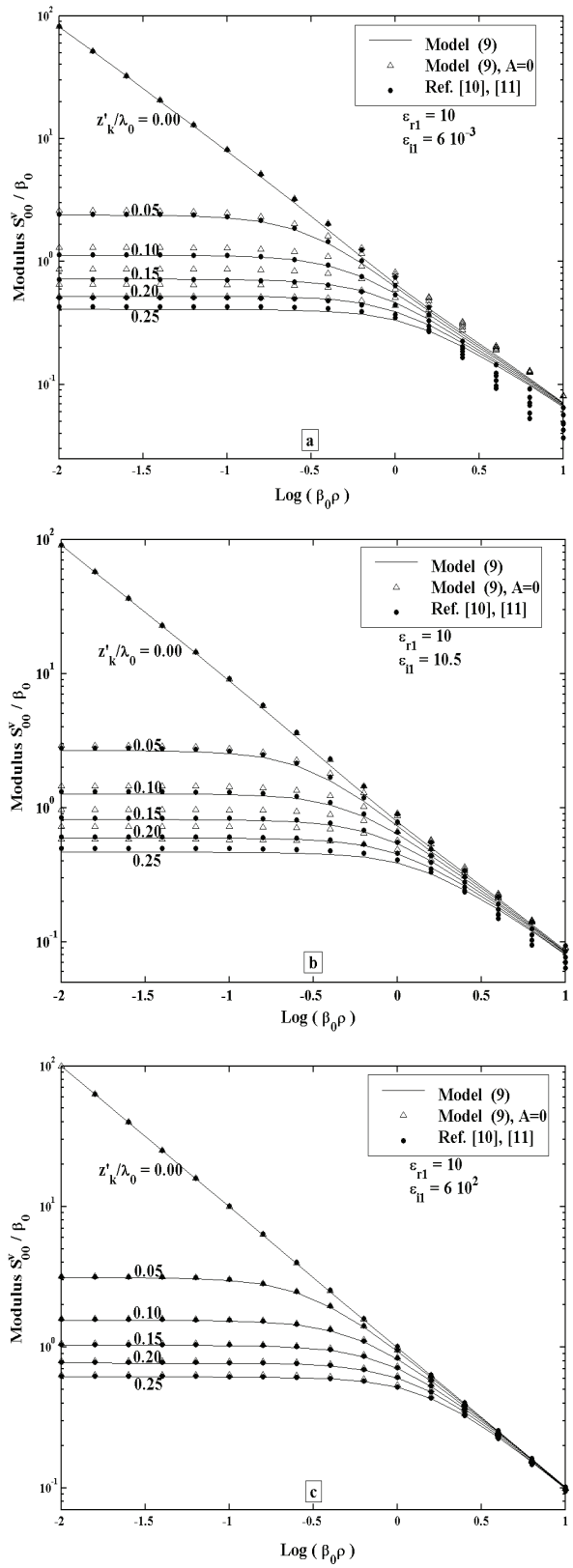

Fig. 3. Modulus of the normalized SIK, at plane $z = 0$, versus normalized radial distance. Relative dielctric constant is $\varepsilon_{r1} = 10$, and VHD position z'_k and imaginary part of dielectric permittivity ε_{i1} are parameters.

Next set of numerical experiments analyses the accuracy of the two-image approximation model given by Eq.(9). Again, the proposed SIK model was tested in the entire range of electrical parameters of the lossy half-space. The results for the modulus of the normalized SIK, S_{00}^{v}/β_0, versus normalized radial distance are presented in Figs.3a, b and c. As previously, parameters are the VHD position z_k' and electrical parameters of the ground. Parallel, the results obtained applying the SIK model given by Eq.(9) when $A=0$, are also given in figures (open up triangle). Comparing was performed using the accurate results from [10] (and [11], solid circle).

IV. CONCLUSION

Two different models for the SIK evaluation in the surroundings of the VHD placed above a lossy half-space are analysed in the this paper. Based on the presented numerical results, one can conclude that both models are, besides of simple form, also satisfyingly accurate. Also, the models can be characterized as general, since no limitations regarding the electrical parameters of the ground were introduced during their derivation. Since the SIK can be easily and very accurately solved using described approximations, the usage of all program tools developed for characterization of vertical wire structures placed above an ideally conducting ground is possible with only minor corrections needing to be done.

This implicates the possibility to completely characterize EM field structure of various vertical wire structures (e.g. vertical dipole antenna, vertical mast antenna, vertical coupled antennas, YAGI structure, etc.) in its near, and using the reciprocity theorem, also approximately in the far field zone. Also, proposed models can be very successfully used for modelling EM field structure of atmospheric discharge ([15]).

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