Investigation of Maximally Flat Fractional Delay All-pass Digital Filters

Kamelia S. Ivanova[1] and Georgi K. Stoyanov[2]

Abstract – **In this paper the relations between the allpass transfer function poles placement and the fractional delay parameter values are analysed and new closed form expressions are derived. It is shown that the poles are taking very unusual positions compared to other filter realizations. Then, the sensitivities of the most popular allpass sections are investigated and the most appropriate structures for different delay-time values are identified. Using these results it is possible to design high accuracy fractional delay structures over different frequency ranges and in a limited wordlength environment.**

Keywords – **IIR digital filters, allpass sections, fractional delay, maximally flat approximation, poles position, low sensitivity.**

I. INTRODUCTION

Recently, there is a growing interest in developing fractional delay digital filters, which appeared to be very useful in numerous fields of digital signal processing and digital communications (timing adjustment, jitter elimination, digital modems, and speech synthesis) [1].

The theory and the design methods of FIR fractional delay filters are quite well developed [1][2][3] and mature enough to have convenient structures to implement them. There are, however, very few publications about IIR fractional delay filters, probably because of the problems connected to the IIR realizations like possible instabilities, higher level of the round-off noises and worst behavior in a limited wordlength environment due to their higher sensitivities. In general the obtained solution has to be checked so that all poles of the filter remain within the unit cycle in the z-domain. The design of IIR fractional delay is by far more complicated than that of corresponding FIR filters. In this work we choose to investigate allpass based fractional delay IIR filters because they have the best magnitude properties, permitting us to concentrate on the phase response characteristics. We use the approximation procedures proposed by Thiran [4], which appear to be the most appropriate for the design of fractional delay digital structures with a maximally flat group delay.

When designing recursive digital filters in limited wordlength environment, it is very important to develop or choose allpass sections with minimized sensitivities for every given transfer function poles position. But the pole-positions are varying considerably for different values of the realized fractional delay, so a thorough analysis of the connections between the poles placement and the fractional delay parameter values has been made in this paper. Additionally, we investigate the range of fractional delay parameter values for which the allpass sections are remaining stable as well as the range of values for which the allpass sections have only real poles. We derive analytical relations between the fractional delay parameter values and the poles for second, third and fourth order allpass sections.

The results so obtained are presented analytically and graphically. These results generalize the behavior of the fractional delay allpass sections so they can be used to design high accuracy fractional delay structures in a different frequency range and in a limited wordlength environment.

II. ANALYSES OF ALLPASS BASED FRACTIONAL DELAY FILTERS OF DIFFERENT ORDER

There are several approaches to approximate given phase, group delay, or phase delay response specifications [1][2][3]. To obtain maximally flat group delay responses, we select the method proposed by Thiran because it provides a closed form solution for allpass transfer function coefficients. The coefficients of an allpass filter with a maximally flat group delay response at the zero frequency can be expressed as [4]:

$$a_k = (-1)^k \binom{N}{k} \prod_{n=0}^{N} \frac{D-N+n}{D-N+k+n}, \; for \; k = 0, 1, 2 \ldots N. \quad (1)$$

This allpass filter is stable when $D > N$ and when $N-1 < D < N$ as it was observed in [2]. We have shown in [7] that the transfer function pole placements are closely related to the fractional delay parameter values. The fractional delay parameter values must be very carefully selected to keep the transfer function poles position inside of the unit circle.

A. Investigation of a second order transfer function

It is easy to obtain the two real poles of the second order fractional delay allpass transfer function when fractional delay parameter value is $1 < D < 2$ and the pair of complex-conjugate decision when $D > 2$. The complex-conjugate poles pair can be expressed as a function of the fractional delay parameter value as follows

[1]Kamelia S. Ivanova is with the Faculty of Communications and Communications Technology, Technical University, Kliment Ohridski 8, 1000 Sofia, Bulgaria, E-mail: ksi@tu-sofia.bg

[2]Georgi K. Stoyanov is with the Faculty of Communications and Communications Technology, Technical University, Kliment Ohridski 8, 1000 Sofia, Bulgaria, E-mail:stoyanov@ieee.org

$$p_{1,2} = \frac{D-2}{D+1} \pm j\sqrt{\frac{3(D-2)}{(D+1)^2(D+2)}} \tag{2}$$

The possible poles positions as a function of increasing fractional delay parameter values from two to infinity are shown in Fig. 1. One could notice that transfer function poles occupy fixed position on the root loci.

The most common requirement for real applications is time delay with small fractional delay parameter values ($N-0.5 < D < N+0.5$, where N is the transfer function order) which means that poles should be positioned near $z=0$ (more specifically in the range between 0 and 0.2 on both real and imaginary axes in the z plane).

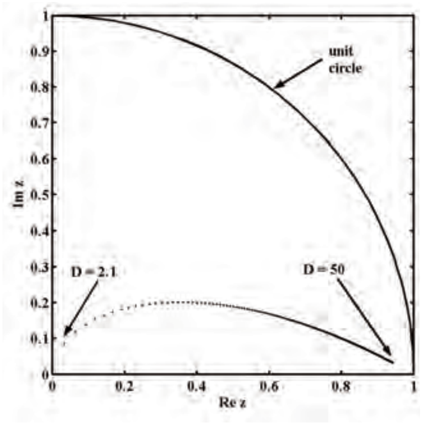

Fig. 1. Possible pole positions of second order allpass transfer function

B. Investigation of a third order transfer function

Similar investigation can be made for third order allpass transfer function. Third order fractional delay allpass filter is stable for fractional delay parameter values $D > 2$. In most of the cases there are one real and a pair of complex-conjugated poles. We identify two distinct situations. In the first one, for $2 < D < 3$, the real pole is negative and the complex-conjugated poles are with positive real parts on the lower root loci. This placement is specific for $N-0.5 < D < N$ fractional delay parameter values. In the second case, for $D > 3$, the real pole and the real part of the complex-conjugated pair are positive. The complex-conjugated poles take values on the upper root loci (Fig. 2). Here one could conclude that poles placement for small fractional delay parameter values are concentrated in the vicinity of $z=0$.

$$p_1 = \frac{\sqrt[3]{C}}{A(D+1)} - 4\frac{(D-3)A}{(D+1)(D+2)\sqrt[3]{C}} + \frac{D-3}{D+1}, \tag{3}$$

$$p_{2,3} = -\frac{\sqrt[3]{C}}{2A(D+1)} + 2\frac{(D-3)A}{(D+1)(D+2)\sqrt[3]{C}} + \frac{D-3}{D+1} \pm j\frac{\sqrt{3}}{2}\left(\frac{\sqrt[3]{C}}{A(D+1)} + 4\frac{(D-3)A}{(D+1)(D+2)\sqrt[3]{C}}\right), \tag{4}$$

where

$$A = D^3 + 6D^2 + 11D + 6, \tag{5}$$

$$B = \frac{D^3 - 8D^2 + 21D - 18}{D+2} \text{ and} \tag{6}$$

$$C = (-4D^2 + 40D - 84 + \sqrt{45B}D + \sqrt{45B})A^2(D+1). \tag{7}$$

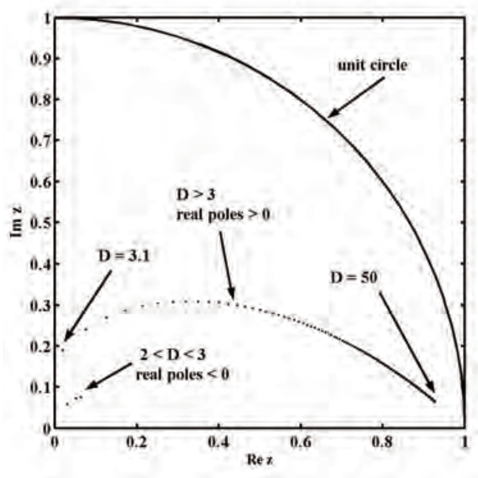

Fig. 2. Possible pole positions of third order allpass transfer function

C. Investigation of a fourth order transfer function

Investigation of fourth order transfer functions leads to similar conclusions, as shown in Fig. 3. One specific distinction of this function is that the upper loci have negative real part for small variation of fractional delay parameter values *D*.

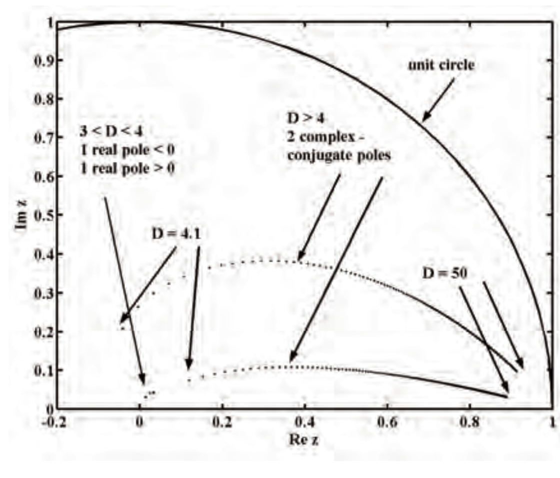

Fig. 3. Possible pole positions of fourth order allpass transfer function

At this point it is easy to make a generalized conclusion for the behavior of *N-th* order allpass structures: they are stable for $D > N-1$, given that for $D = N$ there exist *N* solutions in $z=0$. There are always real poles for the range of values $N-1 < D < N$, and at least one of them is always negative. There are always pairs of complex-conjugated poles for the case of $D > N$, and increasing of *N* leads a shift for the bigger part of the loci toward the left half of the *z*-plane.

III. SECOND ORDER FRACTIONAL DELAY ALLPASS SECTIONS

It is clear from the previous section that the transfer function poles of the allpass circuits with fractional delay are laying in the vicinity of *z*=0 and thus we need realizations with higher pole-density in this region in order to ensure high fractional delay time accuracy. Our extensive search has shown that no such realizations are existing. The most promising candidate is the one based on the famous coupled form having equal pole-density inside the unit circle. Unfortunately, we could not synthesize an allpass section with uniform pole-distribution and because of that we have to investigate and compare the other known allpass sections in order to identify these with lower sensitivity for each pole-positions.

We have shown in [7] that for small values of fractional delay parameter $N-0.5<D<N+0.5$, the phase delay response remains constant over wider range of frequencies and this range is narrowing when increases. In fact realizations with larger *D* (i.e. transfer function poles near $z=1$) can be used for implementation of fractional delay filters for very special applications.

When we want to achieve larger non-integer time delay, it is recommendable to use a cascade with the necessary integer number of delay elements and one second order fractional delay allpass structure. This will ensure the largest possible frequency range over which the group delay response will stay flat.

The transfer function of the most popular allpass sections are as follows [5]:

$$H_{MH2A}(z)=\frac{b_1b_2-b_1z^{-1}+z^{-2}}{1-b_1z^{-1}+b_1b_2z^{-2}}; \tag{8}$$

$$H_{MH2B}(z)=\frac{b_2-b_1z^{-1}+z^{-2}}{1-b_1z^{-1}+b_2z^{-2}}; \tag{9}$$

$$H_{KW2A}(z)=\frac{1+a_1-a_2-(a_1+a_2)z^{-1}+z^{-2}}{1-(a_1+a_2)z^{-1}+(1+a_1-a_2)z^{-2}}; \tag{10}$$

$$H_{KW2B}(z)=\frac{d_1+d_2-1-(d_1-d_2)z^{-1}+z^{-2}}{1-(d_1-d_2)z^{-1}+(d_1+d_2-1)z^{-2}}; \tag{11}$$

$$H_{GM2}(z)=\frac{-a_1-a_2(1-a_1)z^{-1}+z^{-2}}{1-a_2(1-a_1)z^{-1}-a_1z^{-2}}; \tag{12}$$

$$H_{AL}(z)=\frac{-a_1+a_2(1-a_1)z^{-1}+z^{-2}}{1+a_2(1-a_1)z^{-1}-a_1z^{-2}}; \tag{13}$$

$$H_{ST2A}(z)=\frac{1-2b-2(1-2a-b+2ab)z^{-1}+z^{-2}}{1-2(1-2a-b+2ab)z^{-1}+(1-2b)z^{-2}}; \tag{14}$$

$$H_{ST2B}(z)=\frac{1-c_2+(-2+2c_1+c_2)z^{-1}+z^{-2}}{1+(-2+2c_1+c_2)z_{-1}+(1-c_2)z^{-2}}. \tag{15}$$

After representing the coefficients of these sections with the coefficients of the Thiran approximation (1), we get the results shown in Table 1-4. With these formulae it is possible to design and realize the corresponding allpass sections for any given delay parameter *D*.

TABLE I
MH2A AND MH2B FRACTIONAL DELAY FILTER COEFFICIENTS

MH2A		MH2B	
b_1	b_2	b_1	b_2
$2\frac{(D-2)}{(D+1)}$	$\frac{(D-1)}{2(D+2)}$	$2\frac{(D-2)}{(D+1)}$	$\frac{(D-1)(D-2)}{(D+1)(D+2)}$

TABLE II
KW2A AND KW2B FRACTIONAL DELAY FILTER COEFFICIENTS

KW2A		KW2B	
a_1	a_2	d_1	d_2
$\frac{(D^2-3D-4)}{(D+1)(D+2)}$	$\frac{(D^2+3D-4)}{(D+1)(D+2)}$	$2\frac{(D-1)}{(D+2)}$	$\frac{6}{(D+1)(D+2)}$

TABLE III
AL AND GM2 FRACTIONAL DELAY FILTER COEFFICIENTS

AL		GM2	
a_1	a_2	a_1	a_2
$-\frac{(D-1)(D-2)}{(D+1)(D+2)}$	$-\frac{(D-2)(D+2)}{(D^2+2)}$	$-\frac{(D-1)(D-2)}{(D+1)(D+2)}$	$\frac{(D-2)(D+2)}{(D^2+2)}$

TABLE IV
ST2A AND ST2B FRACTIONAL DELAY FILTER COEFFICIENTS

ST2A		ST2B	
a	b	c_1	c_2
$\frac{3}{D^2+2}$	$\frac{3D}{(D+1)(D+2)}$	$\frac{6}{(D+1)(D+2)}$	$\frac{6D}{(D+1)(D+2)}$

IV. SENSITIVITY INVESTIGATIONS

Next we investigate the phase response sensitivities of the fractional delay allpass sections so obtained for two delay parameter *D* values using the package PANDA [6]. The worst case sensitivities (with transfer function coefficients given in the tables) are shown in Fig. 4 – Fig. 11. Thus it appeared that the Mitra and Hirano (MH2A and MH2B), Gray-Markel (GM2) and Ansari-Liu (AL) structures are the most appropriate for small delay parameter values since they have the lowest phase response sensitivity. For poles near $z=1$

the low sensitivities sections ST2A and ST2B are behaving much better than all other known sections.

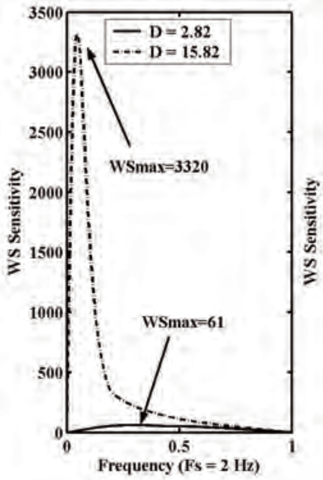

Fig. 4. Worst case sensitivity of MH2A

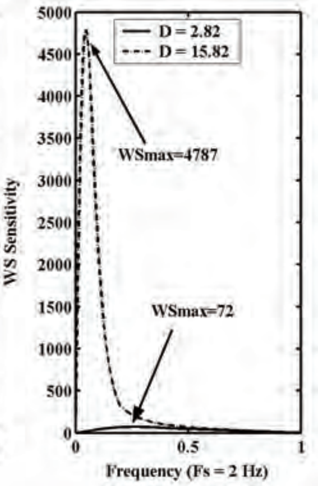

Fig. 5. Worst case sensitivity of MH2B

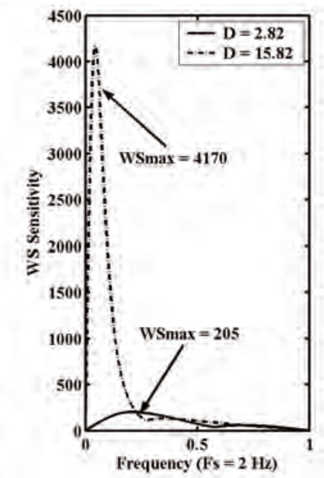

Fig. 6. Worst case sensitivity of KW2A

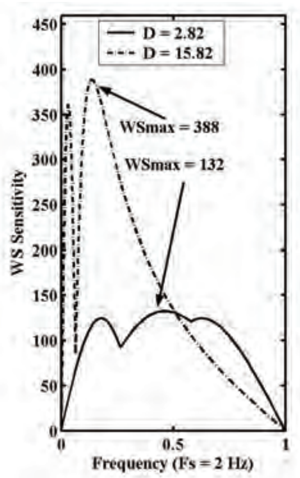

Fig. 7. Worst case sensitivity of KW2B

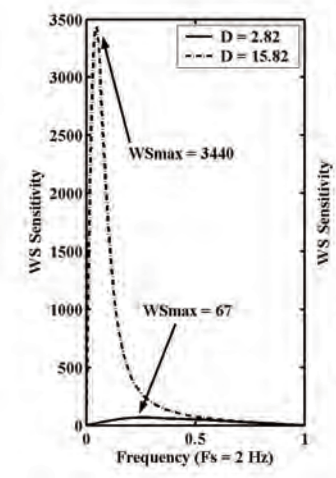

Fig. 8. Worst case sensitivity of AL

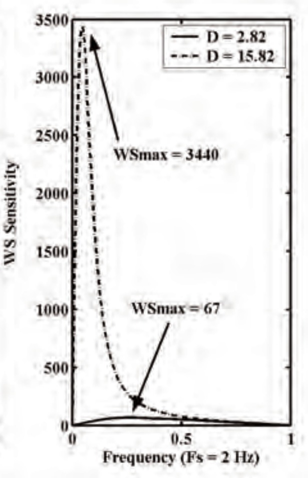

Fig. 9. Worst case sensitivity of GM2

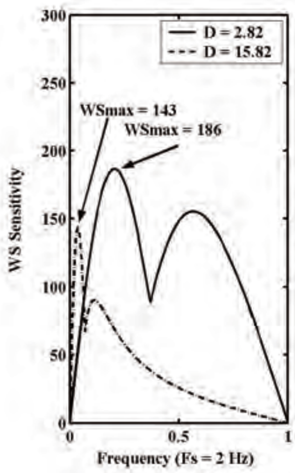

Fig. 10. Worst case sensitivity of ST2A

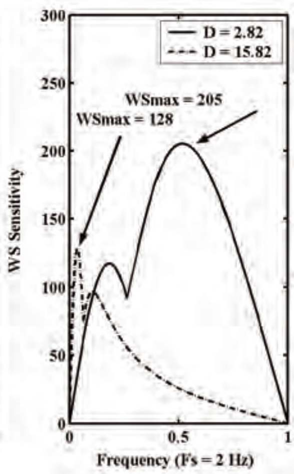

Fig. 11. Worst case sensitivity of ST2B

V. CONCLUSIONS

In this paper we have investigated the behavior of the fractional delay allpass filters with maximally flat group delay response. It was found that transfer function poles of these filters are situated quite differently compared to the ones of the known allpass sections. It was shown that the sections sensitivities depend strongly on the value of the fractional delay parameter *D* and the most suitable sections for some typical pole-locations have been pointed out. Similar sensitivity analysis for other pole-locations (different values of *D*) should be conducted and the most proper allpass sections should be selected or synthesized in order to ensure an accurate realization of the fractional delay in a limited wordlength environment.

REFERENCES

[1] T. Laakso, V. Valimaki, M. Karjalainen and U. Laine, “Splitting the unit delay – tools for fractional delay design”, IEEE Signal Processing Mag., vol. 13, no. 1, pp. 30 – 60, Jan. 1996.

[2] V. Valimaki, “Discrete-Time Modeling of Acoustic Tubes Using Fractional Delay Filters”, Dr. Tech. Thesis, Espoo, Finland; Helsinki Univ. of Tech., Faculty of Electrical Eng., Laboratory of Acoustics and Audio Signal Processing, Dec. 1995.

[3]. V. Valimaki, ”A new filter implementation strategy for Lagrange interpolation”, in Proc. IEEE Int. Symp. Circuits and Systems (ISCAS’95), Seattle, Washington, vol. 1, pp. 361 – 364, April 29–May 3, 1995.

[4] J.–P. Thiran, ”Recursive digital filter with maximally flat group delay”, IEEE Trans. Circuit Theory, vol. 18, no. 6, pp. 659 – 664, Nov. 1971.

[5] G. Stoyanov and A. Nishihara, Very low sensitivity design of digital IIR filters using parallel and cascade-parallel connections of all-pass sections, Bulletin of INCOCSAT, Tokyo Institute of Technology, Vol. 1, pp. 55-64, March 1995.

[6]. N. Sugino and A. Nishihara, “Frequency-domain simulator of digital networks from the structural description”, Trans.of the IEICE of Japan, Vol. E73, No. 11, pp.1804-1806, Nov.1990.

[7] K. S. Ivanova, V. I. Anzova, G. K. Stoyanov, “Low sensitivity realizations of allpass based fractional delay filter”, Proc. TELECOM’2005, St. Konstantine and Elena, Varna, Bulgaria, pp. 376-382, Oct. 7-9, 2005.