Non-uniform Thresholds for Removal of Signal-Dependent Noise in Wavelet Domain

Mitko Kostov, Cvetko Mitrovski[1] and Momcilo Bogdanov[2]

Abstract – **This paper presents experimental results obtained using a method that we propose for signal denoising. Noisy signals are processed in a discrete wavelet transform domain with a non-uniform threshold adjusted to the noise level.**

Keywords – **Denoising, signal-dependent noise, threshold, wavelet domain filtering.**

I. INTRODUCTION

There are many methods for noise removal, but very few of them focus on removing varying noise that depends on the local intensity of the signal. This kind of signal-dependent noise is commonly found in nuclear medicine (NM) images. Until now, the offered methods have been based on conventional filtering in time and frequency domain and lately, wavelet transforms. Research to date in wavelet-domain filtering has focused on removing Gaussian noise by using a global threshold that is independent on the signal or by multiscale products of the detail coefficients [1-3]. These methods are inappropriate for denoising signals that contain signal-dependent noise. One simple fix would be to work with the square-root of the image, since this operation is variance stabilizing [1]. Another method for Poisson noise removal in the wavelet domain uses a non-uniform threshold for filtering the noisy wavelet coefficients [4].

In this paper we present results obtained using our method for removal of signal-dependent noise. It is based on generating non-uniform threshold adjusted to the noise level in the signal. It is organized as follows. The standard wavelet shrinkage program is outlined in Section II. In Section III we discuss how to estimate the varying threshold. In Section IV we verify the validity of our approach on 1-D deterministic signal contaminated with artificially added noise proportional to the signal intensity. At the end, Section V concludes the paper.

II. WAVELET DOMAIN FILTERING

In series expansion of discrete-time function f using wavelets

$$f(t)=\sum_{j=1}^{J}\sum_{k=1}^{2^{-j}M} d_{jk}\psi_{jk}(t)+\sum_{k=1}^{2^{-J}M} a_{Jk}\phi_{Jk}(t)\,, \qquad (1)$$

ψ_{jk} and ϕ_{jk} denote wavelet and scaling function, respectively, the indexes j and k are for dilatation and translation, and a_{Jk} and d_{jk} are approximation and detail coefficients.

The most popular form of wavelet-based filtering, wavelet shrinkage [1], is performed by weighting the corresponding detail wavelet coefficient by h_{ik} $(0 \le h_{jk} \le 1)$ and calculating the inverse wavelet transformation. Conventionally, the filtration is performed either by using "hard threshold" nonlinearity

$$h_{jk}^{(\mathrm{hard})}=\begin{cases}1, & \text{if } |d_{jk}|\ge\tau_j\\ 0, & \text{if } |d_{jk}|<\tau_j\end{cases} \qquad (2)$$

or by using "soft threshold" nonlinearity

$$h_{jk}^{(\mathrm{soft})}=\begin{cases}1-\dfrac{\tau_j\,\mathrm{sgn}(d_{jk})}{d_{jk}}, & \text{if } |d_{jk}|\ge\tau_j\\ 0, & \text{if } |d_{jk}|<\tau_j\end{cases} \qquad (3)$$

where τ_j is a user-specified threshold level.

III. ESTIMATING NON-UNIFORM THRESHOLD

Let $\mathbf{y}$ denotes a noisy signal that consists of a noise-free signal $\mathbf{s}$ and noise $\mathbf{n}$ with zero mean value and energy proportional to the local signal intensity:

$$\mathbf{y} = \mathbf{s} + \mathbf{n}. \qquad (4)$$

For this signal the wavelet transform (WT) satisfies

$$\mathrm{WT}(\mathbf{y}) = \mathrm{WT}(\mathbf{s}) + \mathrm{WT}(\mathbf{n}). \qquad (5)$$

Let $\mathbf{A}$ and $\mathbf{D}$ denote the approximation and detail wavelet coefficients obtained with wavelet transform of the signal $\mathbf{y}$. Since the noise is proportional to the local signal intensity, a threshold τ_j for filtering of all the detail wavelet coefficients $\mathbf{D}$ should not be uniform, but it should follow the local signal intensity. Hence, a non-uniform threshold could be determined as $\boldsymbol{\tau} = \alpha|\mathbf{A}|$ where α is a constant parameter which could be determined by equalizing the energy of the approximation and the detail coefficients:

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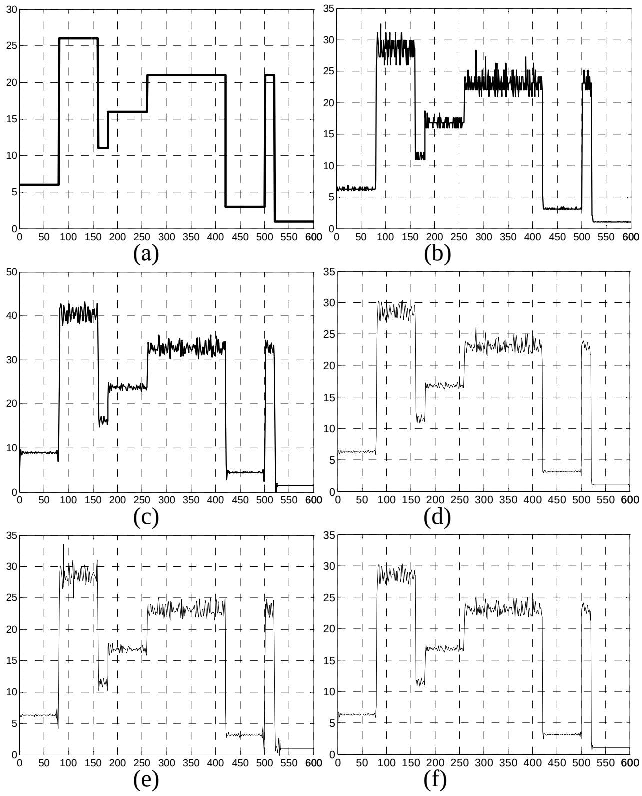

Fig. 1. (a) Deterministic signal; (b) noisy signal; (c) first level approximation coefficients; (d) reconstructed signal using the proposed approach; (e) reconstructed signal using the universal global threshold with db7; (f) reconstructed signal using multiscale product.

$$\sum_i D(i)^2 = \sum_i \left(\alpha A(i)\right)^2 . \quad (6)$$

In this paper we use eq. (6) for the determination of α, based on a set of two new vectors $\mathbf{D_1}$ and $\mathbf{A_1}$ (with lower dimension than the initial vectors $\mathbf{D}$ and $\mathbf{A}$) which are created by using the following.

The detail coefficients $\mathbf{D}$ are like waves and they frequently change their polarity. Therefore, the coefficients between the positive and negative peaks have magnitudes that are close to zero, and we discard their contribution by keeping the local extremes in $\mathbf{D}$ and zeroing the other coefficients:

$$D_1(i) = \begin{cases} D(i) & \text{if} \quad \begin{array}{l} D(i) > \max\left(D(i-1), D(i+1)\right) \\ D(i) < \min\left(D(i-1), D(i+1)\right) \end{array} \\ 0 & \text{otherwise} \end{cases} \quad (7)$$

Similarly, the vector $\mathbf{A_1}$ is constructed by zeroing the approximation coefficients $\mathbf{A}$ for those indices i where $D_1(i) = 0$:

$$\mathbf{A_1} = \mathbf{A} \cdot \mathrm{sign}(|\mathbf{D}|)$$

Since the coefficients $\mathbf{D_1}$ and $\alpha\mathbf{A_1}$ have equal energy and at the same time, the coefficients $\mathbf{D_1}$ contain narrower and higher peaks compared to the coefficients $\alpha\mathbf{A_1}$, the coefficients $\alpha\mathbf{A_1}$ will be smaller than the coefficients $|\mathbf{D_1}|$ where the signal portion in (5) is bigger, but bigger than coefficients $|\mathbf{D_{n1}}|$ where there is no signal.

In general, since the noise is proportional to the local signal intensity, for the threshold $\boldsymbol{\tau}$ the following can be written:

$$\tau(i) = \alpha_n A(i)^n + \cdots + \alpha_1 A(i) + \alpha_0, \quad i = 0, \cdots, L-1, \quad (9)$$

where L is the length of the vectors $\mathbf{A}$ and $\boldsymbol{\tau}$. The coefficients α_0, α_1, … can be obtained by minimizing the square measure E_1 in the smallest squares sense:

$$E_1 = \frac{1}{2} \sum_i \left(|D_1(i)| - \left(\alpha_n A_1(i)^n + \cdots + \alpha_1 A_1(i) + \alpha_0 \right) \right)^2 \quad (10)$$

For the purpose of simplicity, the threshold $\boldsymbol{\tau}$ can take the form (6), and in the same time the error function E_1 which is to be minimized is:

$$E_1 = \frac{1}{2} \sum_i \left(|D_1(i)| - \alpha A_1(i) \right)^2 . \quad (11)$$

IV. Experimental Results

In this Section, we illustrated our proposal on a deterministic 1-D signal contaminated with artificially generated noise in Fig. 1. The non-decimated wavelet transform [3] is performed using a NPR-QMF prototype filter [5], instead of wavelet filters. We obtained that the approximation coefficients follow the signal contour as it is illustrated in Fig. 1c.

Since the noise is signal dependent, the detail coefficients $\mathbf{D}$ (Fig. 2) follow the signal level: the signal intensity in the interval 120-160 is higher than the signal intensity in the intervals 160-180 and 180-220, so the noise is highest in the interval 120-160, while lowest in the interval 180-220.

By comparing Fig. 2a and Fig. 2b it can be seen that the coefficients $\mathbf{D}$ contain signal details $\mathbf{D_s}$ with higher intensities around the positions 160 and 180 (jumps in Fig. 2a, i.e. peaks in Fig. 2b); while in the other regions, in the interval 120-220, there is noise. Also, in Fig 2b it can be noticed that a significant portion of the detail coefficients have values close to zero as a consequence of the fast changing of their polarity.

We experimented with non-uniform thresholds calculated in two ways:

1) with eq. (6) by using energy equalizing of the new vectors A_1 and D_1 in (8);

2) with eq. (9) for different polynomial order n and minimizing of (10).

The thresholds follow the height of the detail peaks as it is illustrated in Fig. 3: the noise level is higher; the thresholds are higher and vice versa. If we make a comparison of two thresholds obtained with 1) and 2), the first threshold is extended and closer to the peaks of the coefficients $|\mathbf{D}|$, which means it is better generated than the second one. This is illustrated in Fig. 3. When a threshold is calculated by using (9), the number of terms (n), have not significant impact on the threshold.

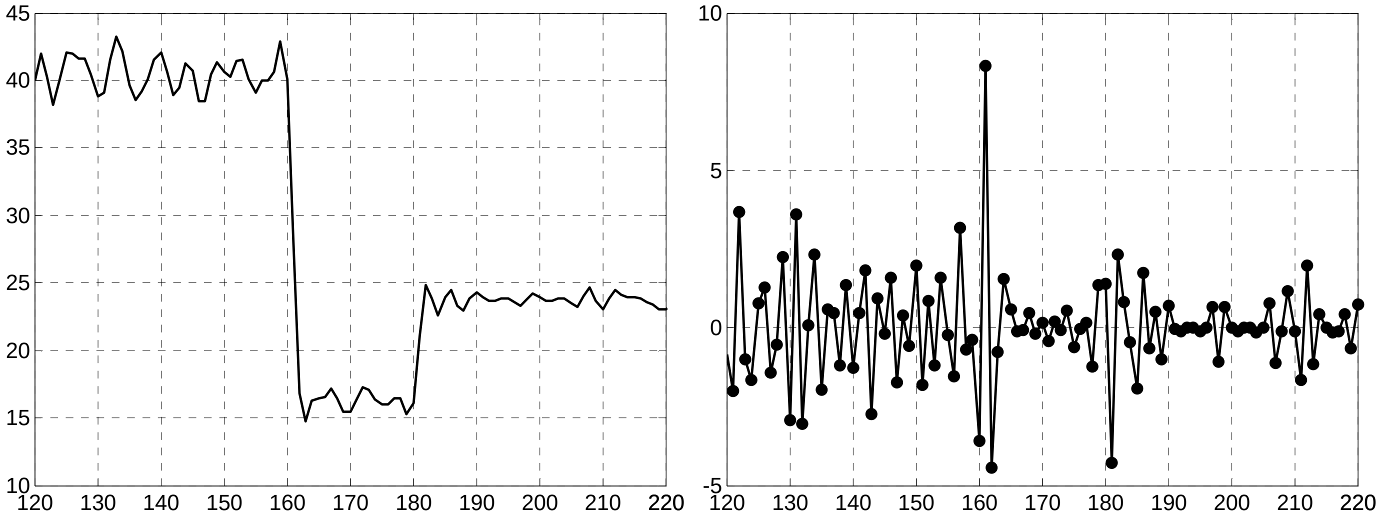

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Fig. 2. A part of the first level wavelet coefficients. (a) Approximation coefficients **A**; (b) detail coefficients **D**.

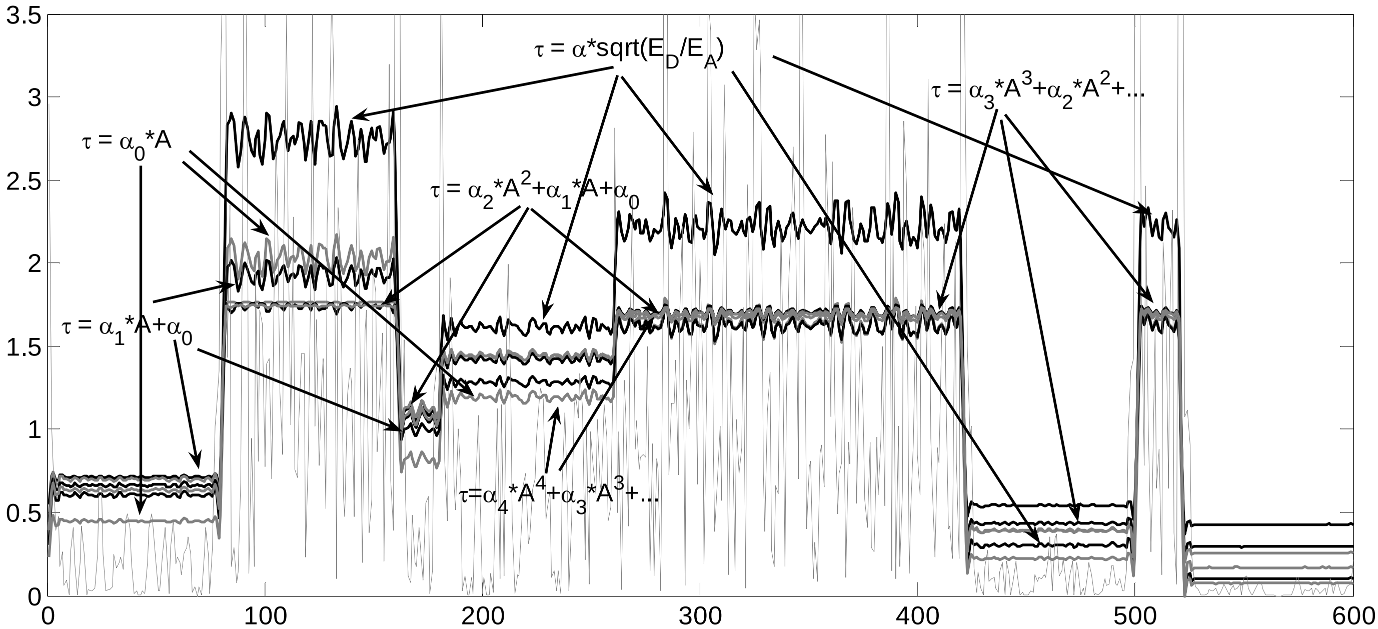


Fig. 3. Details and different estimated non-uniform thresholds: Threshold obtained through energy equalizing when $\boldsymbol{\tau} = \alpha\mathbf{A}$ (8) and thresholds obtained through minimization minimization of (10) for different order of the polynomial (9) (for clearer view, the values from 0 to 3.5 are shown only).

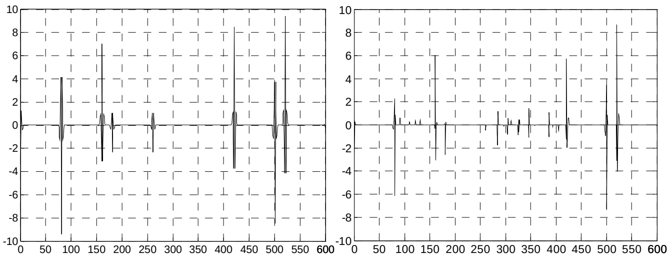
Moreover, the experiment illustrates that although the procedure of energy equalizing is simpler than the procedure of minimizing, it yields with better estimated threshold.

Values of these coefficients α_i in (9) obtained with minimization of (10) in the smallest squares sense and the error (10) for different polynomial order are given in Table I.

Using non-uniform threshold preserve the signal coefficients while remove the noise ones. This is shown in Fig. 4 where noise-free detail coefficients and filtrated detail coefficients are given. The coefficients are filtrated with threshold $\boldsymbol{\tau} = \alpha\mathbf{A}$ where α is estimated through the coefficients $\mathbf{A_1}$ and $\mathbf{D_1}$ in (8). If a global threshold was used, it was not be possible to reduce noise without removing part of the signal at the same time. Hence, the reconstructed signals when a global threshold or multiscale product are used, suffer from distortion at the signal jumps positions, while there is no distortion at the signal filtrated with the proposed approach. This distortion appears as a result of removing signal information contained in the detail coefficients when a global threshold is used. This can be seen from *Fig. 1d, e* and *f*.

In order to quantitatively compare the proposed method to some known wavelet based methods, we use the energy of the remained noise in the filtrated signal $\mathbf{s_1}$ as a measure:

$$E_n = \sum_i \left(s(i) - s_1(i)\right)^2 \tag{12}$$



(a) (b)

Fig. 4. (a) First level detail coefficients of the noise-free signal; (b) Filtrated coefficients using the threshold $\boldsymbol{\tau} = \alpha\mathbf{A}$ when α is estimated through coefficients $\mathbf{A_1}$ and $\mathbf{D_1}$ in (8).

Table 2 contains the results when the signal is reconstructed from the first level approximation and filtrated detail coefficients. It can be seen that when the proposed approach is applied, the noise energy is smaller compared to the other methods that use a global threshold.

V. Conclusion

In this paper we give some views and experimental results conducted with our proposed method for denoising signals that contain signal-dependent noise. The experiments give advantage to the proposed method over the known wavelet based methods.

TABLE I
COEFFICIENTS α_i IN (9) OBTAINED BY MINIMIZATION OF (10) AND THE ERROR (10)

Polynomial order n	Coefficients α_n, α_{n-1}, …, α_1, α_0	Error E in (10)
1 ($\alpha_0 \neq 0$)	0.3714 0.0385	576.2
1 ($\alpha_0 = 0$)	0.0502	585.0
2	0.0484 0.0818 –0.0010	571.1
3	0.2294 0.0422 0.0010 –0.0000	570.9
4	0.2036 0.0315 0.0027 –0.0001 0.0000	571.2
5	0.0643 0.0387 0.0113 –0.0010 0.0000 –0.0000	566.5
6	0.1281 0.0077 0.0043 0.0010 –0.0001 0.0000 –0.0000	560.0
7	0.0035 0.0009 0.0003 0.0001 0.0000 –0.0000 0.0000 –0.0000	600.9
8	1.0e-003 * 0.1347 0.0408 0.0141 0.0047 0.0013 0.0002 –0.0000 –0.0000 0.0000	627.1
9	1.0e-004 * 0.6515 0.2191 0.0848 0.0329 0.0117 0.0033 0.0005 –0.0001 0.0000 –0.0000	607.8
10	1.0e-005 * 0.5918 0.2146 0.0901 0.0387 0.0159 0.0058 0.0017 0.0003 –0.0000 0.0000 –0.0000	634.2

TABLE II
THE ENERGY OF THE REMAINED NOISE IN THE FILTRATED SIGNAL WITH THE PROPOSED METHOD AND KNOWN METHODS

Proposed	Known methods				
	wavelet	universal [1]	[4]	sq.root [1]	[2]
1586	sym4	2011	2032	2071	1566
	sym7	2041	2094	2149	1714
	coif3	1958	1964	2076	1753
	coif5	1923	1959	2059	1847
	db2	2017	2040	2091	1614
	db8	1789	1795	1838	1787

References

[1] D. L. Donoho, "Wavelet Thresholding and W.V.D.: A 10-minute Tour", *Int. Conf. on Wavelets and Applications*, Toulouse, France, June 1992;

[2] Y. Xu, J. B. Weaver, D. M. Healy, Jr., and J. Lu “Wavelet Transform Domain Filters: A Spatially Selective Noise Filtration Technique”, *IEEE Trans. on Image Processing*, vol. 3, no. 6, pp. 747-758, Nov. 1994;

[3] E. J. Balster, Y. F. Zheng and R. L. Ewing “Feature-Based Wavelet Shrinkage Algorithm for Image Denoising”, *IEEE Trans. on Image Processing*, vol. 14, no. 12, pp. 2024-2039, Dec. 2005;

[4] R. D. Nowak, R. G. Baraniuk, ”Wavelet-Domain Filtering for Photon Imaging Systems”, *IEEE Trans. on Image Processing*, vol. 8, Iss. 5, p. 666-678, May 1999;

[5] S. Bogdanova, M. Kostov, and M. Bogdanov, “Design of QMF Banks with Reduced Number of Iterations”, *IEEE Int. Conf. on Signal Processing, Application and Technology, ICSPAT ’99*, Orlando, USA, Nov. 1999.