Cascade Synchronization of Chaotic Systems on the Basis of Linear-Nonlinear Decomposition

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Abstract – **In this paper a method for cascade synchronization of three or more chaotic systems is proposed. The method is based on the so called Linear-Nonlinear decomposition of the systems. The advantage of this approach is in the possibility for exact analysis of the stability of the synchronization manifold, because the error systems are always linear. The results of the application of the method on a well known continuous chaotic system (the Chua system) are proposed.**

Keywords –**Chaotic synchronization, Cascade synchronization, Linear-Nonlinear decomposition, One-way coupling.**

I. INTRODUCTION

One of the most specific and at the same time one of the most interesting fields of the nonlinear dynamics is the chaos theory. During the last 17 years (after 1990) a great effort is made in two main trends in the chaos theory - the synchronization of chaotic systems and the control of these systems [1]. The primary practical benefit of the chaotic synchronization is in the fact that this interesting phenomenon can be used in the secure communications to protect (hide) the transmitted information from unautorised access [2].

The cascade synchronization can be considered as a sub-field of the chaos synchronization, by which three or more chaotic systems have to be synchronized by such way that they will evolve identically but at the same time chaotically in the phase space.

Different methods for synchronization of chaotic systems, respectively for cascade synchronization of such systems exist [3]. It is common that because of the fact that all chaotic systems are nonlinear systems, there is no universal synchronization method, which can always be applied and which guarantees the synchronization for a particular system. Therefore the work of searching for new approaches or new modifications of the existing ones, which overcome some of their drawbacks or limitations, continues in the last years.

In this paper the author proposes a modification of the linear-nonlinear decomposition (LND) method for synchronization of chaotic systems, by which in the Slave systems auxiliary driving signal, proportional to the difference function, is lead in. Thus the main limitation of the LND method (only one variant for coupling of the systems, which for most known chaotic systems doesn't guarantee stable synchronization) is overcome by the great variety of auxiliary couplings, which enhance the chance for achieving stable synchronization. At the same time the proposed approach, called *modified linear-nonlinear decomposition* method (MLND), retains the main advantage of the LND method, being the possibility of exact analysis of the stability of the synchronization manifold. This is conditioned by the fact that the difference system (systems), in contrast to all other synchronization methods, is always linear.

The proposed method is previously tested on simple synchronization of two chaotic systems. In this paper the MLND method is applied for achieving serial cascade synchronization of three or more chaotic systems. The results for the Chua chaotic system are proposed.

II. MODIFIED LINEAR-NONLINEAR DECOMPOSITION METHOD FOR CASCADE CHAOTIC SYNCHRONIZATION

A. Cascade synchronization of chaotic systems

The chaotic synchronization is a phenomenon, by which two identical (most frequently) chaotic systems tune up their dynamics to each other and evolve identically in the phase space. This phenomenon can be used in the communications to secure the transmitted information [2]. For more complex communication systems the receiver and transmitter can be arrays of cascade coupled three or more chaotic systems, or there can exist one or more mediator chaotic systems between the transmitter and receiver. This fact conditions the development of the sub-field of cascade chaotic synchronization.

By the most-frequent type of cascade synchronization three or more identical chaotic systems are coupled sequentially and the proper coupling is searched to achieve stable synchronization between the systems. The first system is called *Master* system, the second one - *Slave*1, but it is also a Master system for the next system of the chain, the third system - *Slave*2 and so on. The systems for the most common case of three chaotic systems can be defined as follows:

$$\textit{Master} \quad \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t), \tag{1}$$

$$\textit{Slave1} \quad \dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{f}}(\tilde{\mathbf{x}}, \mathbf{x}, t), \tag{2}$$

$$\textit{Slave2} \quad \dot{\tilde{\tilde{\mathbf{x}}}} = \tilde{\tilde{\mathbf{f}}}(\tilde{\tilde{\mathbf{x}}}, \tilde{\mathbf{x}}, t), \tag{3}$$

where $\mathbf{x} \in \Re^{n_1}$, $\tilde{\mathbf{x}} \in \Re^{n_2}$, $\tilde{\tilde{\mathbf{x}}} \in \Re^{n_3}$ and the initial conditions are $\mathbf{x}(t_0) \neq \tilde{\mathbf{x}}(t_0) \neq \tilde{\tilde{\mathbf{x}}}(t_0)$. For $n_1 = n_2 = n_3$ and

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$\mathbf{f}(\mathbf{x}) = \tilde{\mathbf{f}}(\tilde{\mathbf{x}}) = \tilde{\tilde{\mathbf{f}}}(\tilde{\tilde{\mathbf{x}}})$ the systems (1)-(3) are identical, which is the most common case. The system (2) is in fact a mediocre system for the synchronization of the systems (1) and (3).

For identical systems, which are considered in this paper, it is called that the three systems are synchronized, if:

$$\lim_{t\to\infty}\mathbf{e}_a(t) = 0 \quad \wedge \quad \lim_{t\to\infty}\mathbf{e}_b(t) = 0\,, \tag{4}$$

where

$$\mathbf{e}_a(t) = \mathbf{x}(t,t_0,\mathbf{x}(t_0)) - \tilde{\mathbf{x}}(t,t_0,\tilde{\mathbf{x}}(t_0)), \tag{5}$$

$$\mathbf{e}_b(t) = \tilde{\mathbf{x}}(t,t_0,\tilde{\mathbf{x}}(t_0)) - \tilde{\tilde{\mathbf{x}}}(t,t_0,\tilde{\tilde{\mathbf{x}}}(t_0)) \tag{6}$$

are the difference functions between the solutions of the first and the second (5), and between the second and the third (6) systems.

The eventual synchronization can also be illustrated directly by observing the difference function between the first and the third systems:

$$\mathbf{e}_c(t) = \mathbf{x}(t,t_0,\mathbf{x}(t_0)) - \tilde{\tilde{\mathbf{x}}}(t,t_0,\tilde{\tilde{\mathbf{x}}}(t_0)). \tag{7}$$

The systems (1) - (3) will be synchronized, if:

$$\lim_{t\to\infty}\mathbf{e}_c(t) = 0\,, \tag{8}$$

but in general it is possible the systems (1)-(2) to achieve marginal synchronization, and the systems (2)-(3) - reciprocal to the first marginal synchronization and thus the condition (8) can be fulfilled without the fulfillment of Eq. (4).

B. Modified linear-nonlinear decomposition (MLND) method for cascade chaotic synchronization

One little known decomposition method for synchronization of two identical chaotic systems is the linear-nonlinear decomposition method [4]. The essence of the method is the formal decomposition of the Master system in linear and nonlinear parts:

$$\textit{Master} \quad \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x},t) = \mathbf{A}\mathbf{x}(t) + \mathbf{h}(\mathbf{x}(t),t)\,, \tag{9}$$

where $\mathbf{A}\mathbf{x}(t)$ is the linear part, and $\mathbf{h}(\mathbf{x}(t),t)$ - the nonlinear part of $\mathbf{f}(\mathbf{x},t)$.

Then the Slave system is constructed in such way, that it is driven with the nonlinear part of Eq.(9):

$$\textit{Slave} \quad \dot{\tilde{\mathbf{x}}}(t) = \tilde{\mathbf{f}}(\tilde{\mathbf{x}},\mathbf{x},t) = \mathbf{A}\tilde{\mathbf{x}}(t) + \mathbf{h}(\mathbf{x}(t),t)\,. \tag{10}$$

Subtracting Eq. (10) from Eq. (9) one gets the error (difference) system:

$$\dot{\mathbf{e}}(t) = \dot{\mathbf{x}}(t) - \dot{\tilde{\mathbf{x}}}(t) = \mathbf{A}(\mathbf{x}(t) - \tilde{\mathbf{x}}(t)) = \mathbf{A}\mathbf{e}(t)\,. \tag{11}$$

The eventual synchronization between systems (9) and (10) will be stable, if $\lim_{t\to\infty}\mathbf{e}(t) = 0$, i.e. the point **e**=0 of the error system (11) is stable. Since Eq. (11) is a linear system, this analysis is easy to made (the stability is proved from the sign of the eigenvalues of **A**), which is the main advantage of the LND method.

However this method has one major limitation - it offers only one variant of coupling of the systems (9) and (10). There is no guarantee, that this variant will give stable synchronization. The more variants of coupling available for any synchronization method, the greater the chance for obtaining stable synchronization. The author suggests the addition of a second coupling, proportional to the error function:

$$\textit{Master} \quad \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{h}(\mathbf{x}(t),t)\,, \tag{12}$$

$$\textit{Slave} \quad \dot{\tilde{\mathbf{x}}}(t) = \mathbf{A}\tilde{\mathbf{x}}(t) + \mathbf{h}(\mathbf{x}(t),t) + \alpha\mathbf{E}(\mathbf{x}(t) - \tilde{\mathbf{x}}(t))\,, \tag{13}$$

where α and **E** are the coupling gain and the coupling matrix which defines the exact form of the coupling. Without loss of generality one can choose the so called standard one-way coupling, by which the connecting nonzero element is in the main diagonal of **E**. The error system:

$$\dot{\mathbf{e}}(t) = \dot{\mathbf{x}}(t) - \dot{\tilde{\mathbf{x}}}(t) = \mathbf{A}(\mathbf{x}(t) - \tilde{\mathbf{x}}(t)) - \alpha\mathbf{E}(\mathbf{x}(t) - \tilde{\mathbf{x}}(t)) = (\mathbf{A} - \alpha\mathbf{E})\mathbf{e}(t) \tag{14}$$

is again linear and thus retaining the advantage of the LND method one can now choose between great number of coupling variants.

This concept can be applied for the cascade chaos synchronization. In this paper, without loss of generality, cascade synchronization of three identical chaotic systems is considered. Then the Master, the Slave1 and the Slave2 systems, when the modified linear-nonlinear decomposition coupling is applied, are:

$$\textit{Master} \quad \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{h}(\mathbf{x}(t),t)\,, \tag{15}$$

$$\textit{Slave1} \quad \dot{\tilde{\mathbf{x}}}(t) = \mathbf{A}\tilde{\mathbf{x}}(t) + \mathbf{h}(\mathbf{x}(t),t) + \alpha_1\mathbf{E}_1(\mathbf{x}(t) - \tilde{\mathbf{x}}(t))\,, \tag{16}$$

$$\textit{Slave2} \quad \dot{\tilde{\tilde{\mathbf{x}}}}(t) = \mathbf{A}\tilde{\tilde{\mathbf{x}}}(t) + \mathbf{h}(\mathbf{x}(t),t) + \alpha_2\mathbf{E}_2(\tilde{\mathbf{x}}(t) - \tilde{\tilde{\mathbf{x}}}(t))\,, \tag{17}$$

where in general $\alpha_1 \neq \alpha_2$ and/or $\mathbf{E}_1 \neq \mathbf{E}_2$.

The two error systems are:

$$\dot{\mathbf{e}}_a(t) = \dot{\mathbf{x}}(t) - \dot{\tilde{\mathbf{x}}}(t) = (\mathbf{A} - \alpha_1\mathbf{E}_1)\mathbf{e}_a(t), \tag{18}$$

$$\dot{\mathbf{e}}_b(t) = \dot{\tilde{\mathbf{x}}}(t) - \dot{\tilde{\tilde{\mathbf{x}}}}(t) = (\mathbf{A} - \alpha_2\mathbf{E}_2)\mathbf{e}_b(t)\,. \tag{19}$$

Both systems (18) and (19) are linear, so when designing the two couplings one can easily prove the stability of each of them.

C. Application of the MLND method on particular chaotic systems

Since most of the known chaotic systems are continuous, some 75% of them being of third order, here the results of applying the modified linear-nonlinear decomposition method for cascade synchronization of one of the well known third-order systems are presented.

The model of the Chua's chaotic electronic circuit is described with the following equations:

$$\begin{aligned} \dot{x}_1 &= \sigma\left[x_2 - (1+b)x_1 - f(x_1)\right], \\ \dot{x}_2 &= x_1 - x_2 + x_3, \\ \dot{x}_3 &= -\beta x_2, \end{aligned} \qquad (20)$$

where $\sigma = 10$, $\beta = 14.87$, $b = -0.68$. The only nonlinearity is $f(x_1) = bx_1 + \frac{a-b}{2}\left(|x_1+1| - |x_1-1|\right)$ with $a = -1.27$.

The typical chaotic attractor of the system is shown in Fig.1. The initial conditions are $\mathbf{x}_0 = \begin{bmatrix}0.1 & 0.1 & 0\end{bmatrix}^T$.

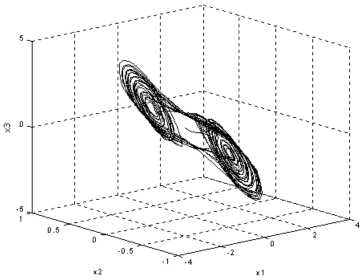

Fig.1. Chua's attractor

If Eq. (20) is considered as a Master system, it can be decomposed in the form of Eq. (9), where:

$$\mathbf{A} = \begin{bmatrix} -\sigma(1+b) & \sigma & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix}, \mathbf{h}(\mathbf{x}(t),t) = f(x_1). \qquad (21)$$

Two of the eigenvalues of **A** are positive - $\lambda_1 = -4.7, \lambda_{2,3} = 0.2 \pm 3.2j$ and the synchronization manifold will be unstable, i.e. the basic LND method cannot be applied neither for plain nor for cascade synchronization.

One of the variants of the MLND method (15)-(17) will be shown. Let the Master system is described with Eq. (12) and the two Slave systems are constructed as follows:

$$Slave1 \quad \begin{aligned} \dot{\tilde{x}}_1 &= \alpha\left[\tilde{x}_2 - (1+b)\tilde{x}_1 - \underline{f_{nl}(x_1)}\right], \\ \dot{\tilde{x}}_2 &= \tilde{x}_1 - \tilde{x}_2 + \tilde{x}_3 + \underline{\alpha_1(x_2 - \tilde{x}_2)}, \\ \dot{\tilde{x}}_3 &= -\beta\tilde{x}_2. \end{aligned} \qquad (22)$$

$$Slave2 \quad \begin{aligned} \dot{\tilde{\tilde{x}}}_1 &= \alpha\left[\tilde{\tilde{x}}_2 - (1+b)\tilde{\tilde{x}}_1 - \underline{f_{nl}(x_1)}\right] + \underline{\alpha_2(\tilde{x}_1 - \tilde{\tilde{x}}_1)}, \\ \dot{\tilde{\tilde{x}}}_2 &= \tilde{\tilde{x}}_1 - \tilde{\tilde{x}}_2 + \tilde{\tilde{x}}_3, \\ \dot{\tilde{\tilde{x}}}_3 &= -\beta\tilde{\tilde{x}}_2. \end{aligned} \qquad (23)$$

The additional coupling between Master and Slave1 systems is obtained by applying the second variant of the standard one-way coupling (OW2) with $\alpha_1 = 10$. The additional coupling between the Slave1 and Slave2 systems is OW1 with $\alpha_2 = 10$.

The matrixes of the linear error systems (18) and (19) for the chosen coupling schemes are:

$$(\mathbf{A} - \alpha_1\mathbf{E}_1) = \begin{bmatrix} -\sigma(1+b) & \sigma & 0 \\ 1 & -1-\underline{\alpha_1} & 1 \\ 0 & -\beta & 0 \end{bmatrix}, \qquad (24)$$

$$(\mathbf{A} - \alpha_2\mathbf{E}_2) = \begin{bmatrix} -\sigma(1+b) - \underline{\alpha_2} & \sigma & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix}, \qquad (25)$$

with the corresponding eigenvalues:

$$eig(\mathbf{A} - \alpha_1\mathbf{E}_1) = -10.9, -1.6 \pm 1.3j, \qquad (26)$$

$$eig(\mathbf{A} - \alpha_2\mathbf{E}_2) = -13.9, -0.1 \pm 3.8j. \qquad (27)$$

Since all real parts of the eigenvalues (26) and (27) are negative, the necessary conditions for the synchronization stability between each pair of systems are fulfilled. The simulation with Matlab/Simulink confirms the synchronization. The errors $e_{ia} = x_i - \tilde{x}_i$ and $e_{ib} = \tilde{x}_i - \tilde{\tilde{x}}_i$ are shown on Fig.2.

After a period of approximately 30 seconds the three systems, started with different initial conditions each, are completely synchronized. At the same time the chaotic nature of the systems' evolution is retained.

One can confirm the cascade synchronization also by observing the error functions between the Master and the Slave2 systems - $e_{ic} = x_i - \tilde{\tilde{x}}_i$, or by viewing the evolution in the state space (e_{1c}, e_{2c}, e_{3c}). The latter is shown on Fig.3. After the transient period the error system between the first and the third system is stabilized into the origin $(0,0,0)$, i.e. there is identical synchronization between the first and the third system. Generally it is arguable if this automatically means there is also identical synchronization between the first and the second, and between the second and the third chaotic systems. In some cases it is possible the first pair of systems to exhibit marginal synchronization, where the error after achieving synchronization is a nonzero constant, which depends on the initial conditions. Very unlikely, but not impossible in general, the second pair of chaotic systems can also exhibit marginal synchronization, where the error stabilizes in the same constant with different sign. Then the Master and the Slave2 systems will exhibit identical synchronization without the presence of identical synchronization between Master and Slave1, and between the Slave1 and Slave2 systems.

The influence of the coupling gain α_i is also investigated. By OW1 and OW2 couplings for the Chua system the increasing of α_i leads to decreasing in the transient process, e.g. for $\alpha_1 = \alpha_2 = 20$ the cascade synchronization is almost two times faster. However the conclusion about the influence of the coupling gain cannot be made in general. For other chaotic systems or even for the OW3 coupling for the Chua system, the increasing of α_i leads to the loss of synchronization. It is also not recommendable to choose very large gain constant, because this will increase the influence of the eventual noise which is always present in the synchronization channel and therefore is included by the couplings in the Slave systems.

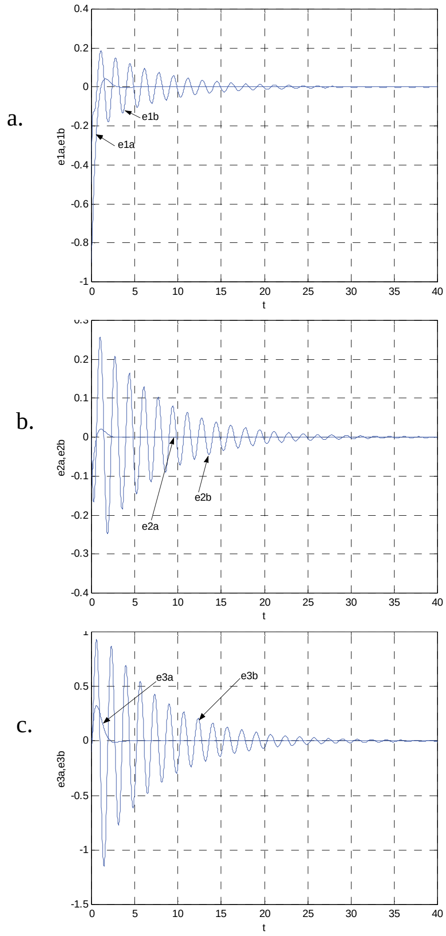

Fig.2. Error functions: a - e_{1a}, e_{1b} ; b - e_{2a}, e_{2b} ; c - e_{3a}, e_{3b}

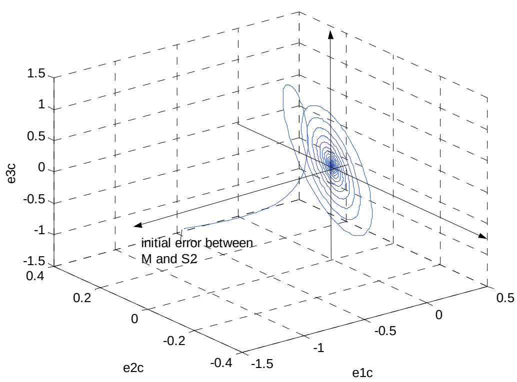

Fig.3. State space $\left(e_{1c}, e_{2c}, e_{3c}\right)$

The results for the other possible variants of the MLND method for the Chua system are generalized in Table I. The columns show the additional coupling between the Master and the Slave 1 systems, the rows show the additional coupling between the Slave 1 and the Slave 2 systems. The coupling constants for all OW1 and OW2 couplings are $\alpha_i = 10$. The coupling constant for the OW3 couplings is $\alpha_i = 2$, because for greater value the error system becomes unstable. The length of the transient before complete identical synchronization between the three systems in simulation seconds is shown. One can see that as long as the basic LND method does not work for the Chua system, all nine possible couplings of the proposed MLND method guarantee stable synchronization. This conclusion however cannot be generalized for all chaotic systems, since each such system as a nonlinear system have its own properties and until now no universal method for chaotic synchronization is proposed.

TABLE I
RESULTS FOR DIFFERENT COUPLINGS

M-S1 / **S1-S2**	OW1	OW2	OW3
OW1	45s	30s	28s
OW2	32s	5s	9s
OW3	30s	7s	10s

The MLND method is also tested on other chaotic systems. For example for the Rossler system the LND method again does not yield stable synchronization while the OW1 and OW2 additional couplings of the MLND method with properly chosen coupling gains stabilize the error systems and the three Rossler systems exhibit identical synchronization. However the OW3 additional coupling for the Rossler system cannot stabilize the error system for all possible coupling gains, so synchronization is not possible.

III. CONCLUSION

In this paper a new modification of the linear-nonlinear decomposition method was presented by which the standard method is combined with additional coupling, proportional to the error function. Thus retaining the main advantage of the LND method, being the possibility for exact stability analysis, one can choose between different types of couplings so the possibility of finding proper synchronization scheme increases.

REFERENCES

[1] Boccaletti, S., J. Kurths, G. Osipov, D. Valladares, C. Zhou. The synchronization of chaotic systems. Physics Reports 366 (2002), pp.1-101.
[2] Carroll, T. Noise-robust synchronized chaotic communications. IEEE Transactions on Circuits and Systems-I, Vol.48, No.12, 2001, pp.1519-1522.
[3] Pecora, L., T. Carroll, G. Johnson, D. Mar, J. Heagy. Fundamentals of synchronization in chaotic systems, concepts, and applications. Chaos 7(4), 1997, pp.520-543.
[4] Yu, H., L. Yanzhu. Chaotic synchronization based on stability criterion of linear systems. Physics Letters A preprint (2003), pp.1-7.