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Some Discretizing Problems in Control Theory

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Abstract – **The methods of obtaining the discrete equivalents for the models of continuous-time objects without and with time delay, as well as some model conversion algorithms, are well-known in the literature. The discretizing and conversing method, presented in this paper, illustrates the use of VAN LOAN's formula for derivation of block triangular matrix exponential [1].**

Keywords - **Control engineering, discretizing problems, matrix exponential, system with time delay, model conversion.**

I. INTRODUCTION

In numerous control applications it is usefull to be able to find the matrix exponential in an effective manner. Moreover, computing integrals involving the matrix exponential is necessary, in order to find the cost equivalents in optimal control theory, for example. Notice, that Van Loan's method [1] for computing four characteristic integrals, based on the derivation of block triangular matrix exponential can be in parctice in control theory. Matrix exponential is anyhow one of the most frequent computed matrix function [2], and many algorithams, developed for that purpose up to now, have bad numerical performances [3].

A method for computing the exponential of a certain block triangular matrix, due to Van Loan [1], is given as follows:

Theorem1. Let n_i, $i = 1,2,3,4$ be positive integers and set m to be their sum. If the $m \times m$ block triangular matrix $\mathbf{C}$ is defined by

$$\mathbf{C} = \begin{array}{c} \left[\begin{array}{cccc} \mathbf{A}_1 & \mathbf{B}_1 & \mathbf{C}_1 & \mathbf{D}_1 \\ \mathbf{0} & \mathbf{A}_2 & \mathbf{B}_2 & \mathbf{C}_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_3 & \mathbf{B}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_4 \end{array}\right] \begin{array}{l} \}n_1 \\ \}n_2 \\ \}n_3 \\ \}n_4 \end{array} \\ \begin{array}{cccc} \underbrace{}_{n_1} & \underbrace{}_{n_2} & \underbrace{}_{n_3} & \underbrace{}_{n_4} \end{array} \end{array}, \tag{1}$$

then for $t \geq 0$

$$e^{\mathbf{C}t} = \begin{bmatrix} \mathbf{F}_1(t) & \mathbf{G}_1(t) & \mathbf{H}_1(t) & \mathbf{K}_1(t) \\ \mathbf{0} & \mathbf{F}_2(t) & \mathbf{G}_2(t) & \mathbf{H}_2(t) \\ \mathbf{0} & \mathbf{0} & \mathbf{F}_3(t) & \mathbf{G}_3(t) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{F}_4(t) \end{bmatrix}, \tag{2}$$

where

$$\mathbf{F}_j(t) = e^{\mathbf{A}_j t}, \quad j = 1,2,3,4 \tag{3}$$

$$\mathbf{G}_j(t) = \int_0^t e^{\mathbf{A}_j(t-s)} \mathbf{B}_j e^{\mathbf{A}_{j+1}s} \,\mathrm{d}s, \quad j = 1,2,3 \tag{4}$$

$$\mathbf{H}_j(t) = \int_0^t e^{\mathbf{A}_j(t-s)} \mathbf{C}_j e^{\mathbf{A}_{j+1}(s-r)} \,\mathrm{d}s + \int_0^t \int_0^s e^{\mathbf{A}_j(t-s)} \mathbf{B}_j e^{\mathbf{A}_{j+1}(s-r)} \mathbf{B}_{j+1} e^{\mathbf{A}_{j+2}r} \,\mathrm{d}r\,\mathrm{d}s, \quad j = 1,2 \tag{5}$$

$$\begin{aligned} \mathbf{K}_1(t) &= \int_0^t e^{\mathbf{A}_1(t-s)} \mathbf{D}_1 e^{\mathbf{A}_4 s} \,\mathrm{d}s \\ &+ \int_0^t \int_0^s e^{\mathbf{A}_1(t-s)} \left[\mathbf{C}_1 e^{\mathbf{A}_3(s-r)} \mathbf{B}_3 + \mathbf{B}_1 e^{\mathbf{A}_2(s-r)} \mathbf{C}_2 \right] e^{\mathbf{A}_4 r} \,\mathrm{d}r\,\mathrm{d}s \\ &+ \int_0^t \int_0^s \int_0^r e^{\mathbf{A}_1(t-s)} \mathbf{B}_1 e^{\mathbf{A}_2(s-r)} \mathbf{B}_2 e^{\mathbf{A}_3(r-w)} \mathbf{B}_3 e^{\mathbf{A}_4 w} \,\mathrm{d}w\,\mathrm{d}r\,\mathrm{d}s\,. \end{aligned} \tag{6}$$

Corollary to Theorem 1. Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{Q}_c$ be real matrices of dimension $n \times n$, $n \times p$, and $n \times n$, respectively. Assume that matrix $\mathbf{Q}_c$ is symmetric $\left(\mathbf{Q}_c^T = \mathbf{Q}_c\right)$ and positive semi-definite $\left(\mathbf{x}^T \mathbf{Q}_c \mathbf{x} \geq 0\right)$. Following the previous theorem and combining various submatrices, it can be shown that the integral

$$\mathbf{Q}(T) = \int_0^T e^{\mathbf{A}^T s} \mathbf{Q}_c e^{\mathbf{A}s} \,\mathrm{d}s \tag{7}$$

can be calculated as

$$\mathbf{Q}(T) = \mathbf{F}_3(T)^T \mathbf{G}_2(T), \tag{8}$$

where

$$\exp\left(\begin{bmatrix} -\mathbf{A}^T & \mathbf{Q}_c \\ \mathbf{0} & \mathbf{A} \end{bmatrix} T \right) = \begin{bmatrix} \mathbf{F}_2(T) & \mathbf{G}_2(T) \\ \mathbf{0} & \mathbf{F}_3(T) \end{bmatrix}. \tag{9}$$

The need for computing intrgral (7) arises in optimal sampled-data regulation problem, for example.

It is posible to compute the exponential of the matrices of low dimension analytically for an arbitary sampling interval T. The advantage of an analytical computation is that the result is expressed in terms of different parameters, and it is possible to examine the effect of changing these parameters. Recall that an arbitrary matrix function $f(\mathbf{C})$ can be computed via the wellknown Cayley-Hamilton Theorem. Moreover, if the matrix $\mathbf{C}$ has distinct eigenvalues, the method of eigenvalue decomposition can be use for computing matrix function [2], [4]. In this paper the problems of discretizing the continuous-time systems without and with time delay, as well as the digital model conversion are solved by computing exponential of some special form matrices.

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II. Discretizing the Models of Analog Plants

For simplicity, without loss of generality, consider the nth-order single-input single-output control object. It is convenient to introduce the realization sets as follows [5]:

$$\mathrm{S_C} \overset{\text{def}}{=} \left\{ \left(\mathbf{A}_\mathrm{C}, \mathbf{b}_\mathrm{C}, \mathbf{d}\right) : G_\mathrm{C}(s) = \frac{N_\mathrm{C}(s)}{D_\mathrm{C}(s)} = \mathbf{d}\left(s\mathbf{I} - \mathbf{A}_\mathrm{C}\right)^{-1}\mathbf{b}_\mathrm{C} \right\}, \tag{10}$$

$$\mathrm{S_q} \overset{\text{def}}{=} \left\{ \left(\mathbf{A}_\mathrm{q}, \mathbf{b}_\mathrm{q}, \mathbf{d}\right) : G_\mathrm{q}(z) = \frac{N_\mathrm{q}(z)}{D_\mathrm{q}(z)} = \mathbf{d}\left(z\mathbf{I} - \mathbf{A}_\mathrm{q}\right)^{-1}\mathbf{b}_\mathrm{q} \right\}, \tag{11}$$

where

$$\mathbf{A}_\mathrm{q} = \mathbf{\Phi}(T) = \mathrm{e}^{\mathbf{A}_\mathrm{C}T}, \tag{12}$$

$$\mathbf{b}_\mathrm{q} = \int_0^T \mathrm{e}^{\mathbf{A}_\mathrm{C}\tau}\mathbf{b}_\mathrm{C}\, \mathrm{d}\tau , \tag{13}$$

and

$$G_\mathrm{q}(z) = \mathrm{Z}\,\mathrm{L}^{-1}\left\{ \frac{1-\mathrm{e}^{-Ts}}{s} G_\mathrm{C}(s) \right\} . \tag{14}$$

Thus, (11) represents *ZOH* equivalent model for (10) at sampling interval T, that is the so called q-model for the continuous-time system. Note, that $\mathbf{A}_\mathrm{q}$ is matrix exponent, and the vector $\mathbf{b}_\mathrm{q}$ must be computed by integration as shown in (13). However, it is interesting to note that both $\mathbf{A}_\mathrm{q}$ and $\mathbf{b}_\mathrm{q}$ can be computed simultaneously using a single matrix exponential.

Define $(n+1)\times(n+1)$ block matrix $\mathbf{M}$ as

$$\mathbf{M} = \left[\begin{array}{c:c} \mathbf{A}_\mathrm{C} & \mathbf{b}_\mathrm{C} \\ \hdashline \mathbf{0} & 0 \end{array}\right] \}n , \tag{15}$$

in which the zero in the lower left-hand corner represents an $n-$dimensional zero row-vector. Then, by using Van Loan's formulas (1)-(6) for matrix exponential of $\mathbf{M}T$ can be found

$$\mathrm{e}^{\mathbf{M}T} = \left[\begin{array}{c:c} \mathbf{A}_\mathrm{q} & \mathbf{b}_\mathrm{q} \\ \hdashline \mathbf{0} & 1 \end{array}\right] . \tag{16}$$

Thus, the digital model matrices $\mathbf{A}_\mathrm{q}$ and $\mathbf{b}_\mathrm{q}$ can be computed as follows

$$\left[\mathbf{A}_\mathrm{q} \mid \mathbf{b}_\mathrm{q}\right] = \left[\mathbf{I} \quad \mathbf{0}\right] \cdot \mathrm{e}^{\mathbf{M}T} , \tag{17}$$

whereas

$$\left[\mathbf{A}_\mathrm{C} \mid \mathbf{b}_\mathrm{C}\right] = \left[\mathbf{I} \quad \mathbf{0}\right] \cdot \mathbf{M} . \tag{18}$$

Example 1. We will use the procedure described in this section to compute continuous-time model plant $\left(\mathbf{A}_\mathrm{C}, \mathbf{b}_\mathrm{C}\right)$ simultaneously on the basis *ZOH* equivalent model for the plant given by $\left[\mathbf{A}_\mathrm{q} \mid \mathbf{b}_\mathrm{q}\right] = \begin{bmatrix} 1 & T & T^2/2 \\ 0 & 1 & T \end{bmatrix}$, where T is the sampling period. To use (15)-(18), we create the matrix

$$\mathbf{M}_1 = \mathrm{e}^{\mathbf{M}T} = \begin{bmatrix} \mathbf{A}_\mathrm{q} & \mathbf{b}_\mathrm{q} \\ \mathbf{0} & 1 \end{bmatrix} = \left[\begin{array}{cc:c} 1 & T & T^2/2 \\ 0 & 1 & T \\ \hdashline 0 & 0 & 1 \end{array}\right]$$

that has an eigenvalue $\lambda = 1$ with multiplicity $m = 3$. The matrix $\mathbf{M}_1$ is positive-definite, and the 3×3 matrix $\mathbf{M}$ can be written as the matrix logarithm

$$\mathbf{M} = f\left(\mathbf{M}_1\right) = \left(\ln \mathbf{M}_1\right)/T .$$

To calculate this matrix function, some formulas based on the wellknown Cayley-Hamilton theorem can be used. Namely, the matrix $\mathbf{M}$ can be written as the matrix polynomial of degree 2 as follows

$$\mathbf{M} = \alpha\left(\mathbf{M}_1\right) = \alpha_1\mathbf{M}_1^2 + \alpha_2\mathbf{M}_1 + \alpha_3 ,$$

To compute the coefficients α_i, $i = 1,2,3$ the scalar function $f(\lambda) = (\ln\lambda)/T$, polynomial $\alpha(\lambda) = \alpha_1\lambda^2 + \alpha_2\lambda + \alpha_3$, as well as their first and second derivatives with respect to λ are required. These equations are calculated at the eigenvalue $\lambda = 1$ as follows

$$\begin{array}{ll} f(1) = 0 & \alpha(1) = \alpha_1 + \alpha_2 + \alpha_3 \\ f^{(1)}(1) = 1/T & \alpha^{(1)}(1) = 2\alpha_1 + \alpha_2 \\ f^{(2)}(1) = -1/T & \alpha^{(2)}(1) = 2\alpha_1 \end{array} .$$

When the values of $\alpha_1 = -1/2T$, $\alpha_2 = 2/T$ and $\alpha_3 = -3/2T$ are substituted in, the result is

$$\mathbf{M} = -\frac{1}{2T}\begin{bmatrix} 1 & 2T & 2T^2 \\ 0 & 1 & 2T \\ 0 & 0 & 1 \end{bmatrix} + \frac{2}{T}\begin{bmatrix} 1 & T & T^2/2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} - \frac{3}{2T}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \left[\begin{array}{cc:c} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hdashline 0 & 0 & 0 \end{array}\right]$$

Finally, $\mathbf{A}_\mathrm{C}$ and $\mathbf{b}_\mathrm{C}$ are extracted from the just derived matrix $\mathbf{M}$ according to the partitions shown in (15). So, we get state-space model $\left(\mathbf{A}_\mathrm{C}, \mathbf{b}_\mathrm{C}\right)$, with $\mathbf{A}_\mathrm{C} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{b}_\mathrm{C} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ for the considered double integrator plant.

III. Model Conversions

Let an integer N denotes the ratio between the slow sampling period T_S and the fast sampling period T, i.e. $N = T_\mathrm{S}/T$. Let $\left(\mathbf{A}_\mathrm{qS}, \mathbf{b}_\mathrm{qS}, \mathbf{d}\right)$ represents the slow discrete-time model of the corresponding continuous-time model (10). The commonly used matrix continued-fraction method to convert $\left(\mathbf{A}_\mathrm{q}, \mathbf{b}_\mathrm{q}, \mathbf{d}\right)$ to $\left(\mathbf{A}_\mathrm{C}, \mathbf{b}_\mathrm{C}, \mathbf{d}\right)$, for example is [6]:

$$\begin{aligned} \mathbf{A}_\mathrm{C} &= \frac{1}{T}\ln \mathbf{A}_\mathrm{q} \approx \frac{2}{T}\mathbf{F} \\ &\approx \frac{2}{T}\mathbf{F}\left[\mathbf{I}_\mathrm{n} - \frac{4}{15}\mathbf{F}^2\right]\left[\mathbf{I}_\mathrm{n} - \frac{3}{5}\mathbf{F}^2\right]^{-1} \approx \cdots \end{aligned} \tag{19}$$

where

$$\mathbf{F} \overset{\text{def}}{=} \left(\mathbf{A}_\mathrm{q} - \mathbf{I}_\mathrm{n}\right)\left(\mathbf{A}_\mathrm{q} + \mathbf{I}_\mathrm{n}\right)^{-1}. \tag{20}$$

The vector $\mathbf{b}_\mathrm{C}$ can be found by

$$\mathbf{b}_\mathrm{C} = \mathbf{A}_\mathrm{C}\left(\mathbf{A}_\mathrm{q} - \mathbf{I}_\mathrm{n}\right)^{-1}\mathbf{b}_\mathrm{q} . \tag{21}$$

The conversion of the fast-rate digital model $\left(\mathbf{A}_q, \mathbf{b}_q, \mathbf{d}\right)$ to a slow-rate digital model $\left(\mathbf{A}_{qs}, \mathbf{b}_{qs}, \mathbf{d}\right)$ with the slow sampling period T_S can be carried out as follows. According to (21), we have $\mathbf{b}_q = \left(\mathbf{A}_q - \mathbf{I}_n\right)\mathbf{A}_c^{-1}\mathbf{b}_c$, which gives $\mathbf{A}_c^{-1}\mathbf{b}_c = \left(\mathbf{A}_q - \mathbf{I}_n\right)\mathbf{b}_q$, and $\mathbf{b}_{qs} = \left(\mathbf{A}_{qs} - \mathbf{I}_n\right)\mathbf{A}_c^{-1}\mathbf{b}_c$. Thus we obtain $\left(\mathbf{A}_{qs}, \mathbf{b}_{qs}\right)$ from $\left(\mathbf{A}_q, \mathbf{b}_q\right)$ as

$$\mathbf{A}_{qs} = \mathbf{A}_q{}^N \tag{22}$$

and

$$\mathbf{b}_{qs} = \left(\mathbf{A}_{qs} - \mathbf{I}_n\right)\left(\mathbf{A}_q - \mathbf{I}_n\right)^{-1}\mathbf{b}_q . \tag{23}$$

Note that the conversion of the fast-rate digital model to a slow-rate one can be obtain in another way by using relation (16), i.e.

$$e^{\mathbf{M}T_S} = \left[\begin{array}{c|c} \mathbf{A}_{qs} & \mathbf{b}_{qs} \\ \hline \mathbf{0} & 1 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{A}_q & \mathbf{b}_q \\ \hline \mathbf{0} & 1 \end{array}\right]^N . \tag{24}$$

By induction it can be shown that

$$\left[\begin{array}{c|c} \mathbf{A}_q & \mathbf{b}_q \\ \hline \mathbf{0} & 1 \end{array}\right]^N = \left[\begin{array}{c|c} \mathbf{A}_q{}^N & \left(\sum_{i=0}^{N-1} \mathbf{A}_q{}^i\right)\mathbf{b}_q \\ \hline \mathbf{0} & 1 \end{array}\right]. \tag{25}$$

So, the matrices of the T_S model and T model have the relation

$$\mathbf{A}_{qs} = \mathbf{A}_q{}^N , \quad \text{and} \quad \mathbf{b}_{qs} = \left(\sum_{i=0}^{N-1} \mathbf{A}_q{}^i\right)\mathbf{b}_q . \tag{26}$$

IV. Discretizing a Continuous-Time System with Time Delay

Consider single-input single-output continuous-time system with time delay described in state-space by

$$\dot{\mathbf{x}}(t) = \mathbf{A}_c\mathbf{x}(t) + \mathbf{b}_c u(t-\tau) \tag{27}$$

$$c(t) = \mathbf{d}\mathbf{x}(t) \ . \tag{28}$$

It is assumed that the time delay is longer than the sampling period T. Let

$$\tau = (d-1)T + \tau' , \tag{29}$$

where $0 < \tau' \le T$ and $d\ (\ge 1)$ is an integer. Discrete-time transfer functions of systems with a delay that is not an integer times the sampling period are easily obtained by using the modified $\mathcal{Z}$ transform [7]. The discrete-time state-space model of system (27)-(29) is given in literature [8]-[10], [4] by

$$\begin{bmatrix} \mathbf{x}[(k+1)T] \\ u(kT) \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi} & \mathbf{\Gamma}_1 \\ \mathbf{0} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(kT) \\ u[(k-1)T] \end{bmatrix} + \begin{bmatrix} \mathbf{\Gamma}_0 \\ 1 \end{bmatrix} u(kT) \quad \text{for } d = 1, \tag{30}$$

or when $d \ge 2$, the equations are

$$\begin{bmatrix} \mathbf{x}[(k+1)T] \\ u(kT-dT+1) \\ \vdots \\ u(kT-T) \\ u(kT) \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi} & \mathbf{\Gamma}_1 & \mathbf{\Gamma}_0 & 0 & \dots & 0 \\ \mathbf{0} & 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & & \vdots \\ \mathbf{0} & 0 & 0 & 0 & \dots & 1 \\ \mathbf{0} & 0 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(kT) \\ u(kT-dT) \\ \vdots \\ u(kT-2T) \\ u(kT-T) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(kT), \tag{31}$$

where $\mathbf{\Phi} = e^{\mathbf{A}_c T}$

$$\mathbf{\Gamma}_1 = e^{\mathbf{A}_c(T-\tau')} \int_0^{\tau'} e^{\mathbf{A}_c\lambda}\mathbf{b}_c \, d\lambda , \text{ and } \quad \mathbf{\Gamma}_0 = \int_0^{T-\tau'} e^{\mathbf{A}_c\lambda}\mathbf{b}_c \, d\lambda . \tag{32}$$

The output equation is obtained from (28) to be

$$c(kT) = \begin{bmatrix} \mathbf{d} & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(kT) \\ u(kT-dT) \\ \vdots \\ u(kT-T) \end{bmatrix} . \tag{33}$$

Notice that the above equations (30) and (31) contain partitioned matrices. Each zero below the matrix $\mathbf{\Phi}$ in (30) and (31) represents a row vector of n zeros. Recall, that because the signal $u(t)$ is piecewise constant over the sampling interval, the delayed signal $u(t-\tau)$ is also piecewise constant. However, the delayed signal will change between the sampling instants as Fig. 1 visualizes.

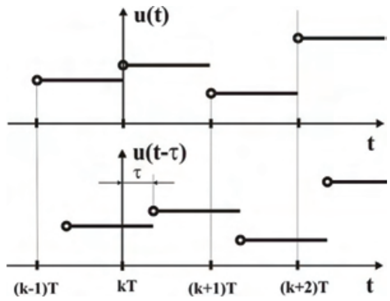

Fig. 1. The piecewise constant signals $u(t)$ and $u(t-\tau)$, $\tau < T$

To integrate the differential equation (27) over one sample period in order to obtain *ZOH* equivalent model, it is convenient to split the integration interval into two parts, so that control signal $u(t-\tau)$ is constant in each part. Hence, the motion of the considered dynamical system (27)-(29) in the interval $kT \le t < (k+1)T$ is

$$\mathbf{x}(t) = \mathbf{\Phi}(t-kT)\mathbf{x}(kT) + \mathbf{\Theta}(t-kT)u[(k-1)T], \quad kT \le t < kT+\tau \tag{34}$$

and

$$\mathbf{x}(t) = \mathbf{\Phi}(t-kT-\tau)\mathbf{x}(kT+\tau) + \mathbf{\Theta}(t-kT-\tau)u(kT), \quad kT+\tau \le t < (k+1)T \tag{35}$$

where

$$\boldsymbol{\Phi}(t) = e^{\mathbf{A}_c t} \quad \text{and} \quad \boldsymbol{\Theta}(t-\lambda) = \int_{\lambda}^{t} \boldsymbol{\Phi}(t-\upsilon)\mathbf{b}_c \, d\upsilon . \qquad (36)$$

If we now substitute $t = kT + \tau$ in (34) and $t = (k+1)T$ in (35), we obtain

$$\mathbf{x}(kT+\tau) = \boldsymbol{\Phi}(\tau)\mathbf{x}(kT) + \boldsymbol{\Theta}(\tau)u\left[(k-1)T\right], \qquad (37)$$

and

$$\mathbf{x}\left[(k+1)T\right] = \boldsymbol{\Phi}(T-\tau)\mathbf{x}(kT+\tau) + \boldsymbol{\Theta}(T-\tau)u(kT) \quad . (38)$$

It is clear, that the relations (37) and (38) can be expressed as the function of the matrices $\mathbf{M}$ and $e^{\mathbf{M}T}$, given by (15) and (16), as shown below:

$$\begin{bmatrix} \mathbf{x}(T+\tau) \\ u(kT-T) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{q1} & \mathbf{b}_{q1} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}(kT) \\ u(kT-T) \end{bmatrix} = e^{\mathbf{M}\tau} \begin{bmatrix} \mathbf{x}(kT) \\ u(kT-T) \end{bmatrix} \qquad (39)$$

and

$$\begin{bmatrix} \mathbf{x}\left[(k+1)T\right] \\ u(kT) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{q2} & \mathbf{b}_{q2} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}(kT+\tau) \\ u(kT) \end{bmatrix} = e^{\mathbf{M}(T-\tau)} \begin{bmatrix} \mathbf{x}(kT+\tau) \\ u(kT) \end{bmatrix} \qquad (40)$$

If we substitute (39) in (40) we obtain

$$\begin{bmatrix} \mathbf{x}\left[(k+1)T\right] \\ u(kT) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{q2} & \mathbf{b}_{q2} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{A}_{q1}\mathbf{x}(kT) + \mathbf{b}_{q1}u(kT-T) \\ u(kT) \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{A}_{q2}\mathbf{A}_{q1}\mathbf{x}(kT) + \mathbf{A}_{q2}\mathbf{b}_{q1}u(kT-T) + \mathbf{b}_{q2}u(kT) \\ u(kT) \end{bmatrix} . \qquad (41)$$

Note that we can compute the product of the matrix exponentials as follows:

$$e^{\mathbf{M}(T-\tau)} e^{\mathbf{M}\tau} = \begin{bmatrix} \mathbf{A}_{q2} & \mathbf{b}_{q2} \\ \mathbf{0} & 1 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{A}_{q1} & \mathbf{b}_{q1} \\ \mathbf{0} & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{A}_{q2}\mathbf{A}_{q1} & \mathbf{A}_{q2}\mathbf{b}_{q1} + \mathbf{b}_{q2} \\ 0 & 1 \end{bmatrix} . \qquad (42)$$

Finally, the equations (41)-(42) can be compared with (30)-(32) resulting in

$$\begin{aligned} \boldsymbol{\Phi} &= \mathbf{A}_{q2}\mathbf{A}_{q1} \\ \boldsymbol{\Gamma}_0 &= \mathbf{b}_{q2} \\ \boldsymbol{\Gamma}_1 &= \mathbf{A}_{q2}\mathbf{b}_{q1} \; . \end{aligned} \qquad (43)$$

Example 2. Calculate the *ZOH* equivalent model for the following continuous-time system with time delay

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t-\tau), \text{ where } \tau = 0.2 \text{ and } T = 0.3 \; .$$

The matrix $\mathbf{M}$ defined in (15) is

$$\mathbf{M} = \begin{bmatrix} \mathbf{A}_c & \mathbf{b}_c \\ \mathbf{0} & 0 \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 1 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right].$$ The matrix exponentials are

$$e^{\mathbf{M}\tau} = \begin{bmatrix} \mathbf{A}_{q1} & \mathbf{b}_{q1} \\ \mathbf{0} & 1 \end{bmatrix} = \left[\begin{array}{cc|c} e^{\tau} & 0 & e^{\tau}-1 \\ \tau e^{\tau} & e^{\tau} & (\tau-1)e^{\tau}+1 \\ \hline 0 & 0 & 1 \end{array}\right],$$

and

$$e^{\mathbf{M}(T-\tau)} = \begin{bmatrix} \mathbf{A}_{q2} & \mathbf{b}_{q2} \\ \mathbf{0} & 1 \end{bmatrix} = \left[\begin{array}{cc|c} e^{T-\tau} & 0 & e^{T-\tau}-1 \\ (T-\tau)e^{T-\tau} & e^{T-\tau} & (T-\tau-1)e^{T-\tau}+1 \\ \hline 0 & 0 & 1 \end{array}\right].$$

Using (43) we get

$$\boldsymbol{\Phi} = \mathbf{A}_{q2}\mathbf{A}_{q1} = \begin{bmatrix} e^{T} & 0 \\ T e^{T} & e^{T} \end{bmatrix} = \begin{bmatrix} 1.350 & 0 \\ 0.405 & 1.350 \end{bmatrix},$$

$$\boldsymbol{\Gamma}_0 = \mathbf{b}_{q2} = \begin{bmatrix} e^{T-\tau}-1 \\ (T-\tau-1)e^{T-\tau}+1 \end{bmatrix} = \begin{bmatrix} 0.105 \\ 0.005 \end{bmatrix},$$

and

$$\boldsymbol{\Gamma}_1 = \mathbf{A}_{q2}\mathbf{b}_{q1} = \begin{bmatrix} e^{T-\tau}(e^{\tau}-1) \\ e^{T-\tau}(1-T+\tau) + e^{T}(T-1) \end{bmatrix} = \begin{bmatrix} 0.245 \\ 0.050 \end{bmatrix} .$$

V. Conclusion

This paper deals with a procedure for simultaneous computing the both matrices of zero-order hold equivalent q-model $\left(\mathbf{A}_q \text{ and } \mathbf{b}_q\right)$ using a single matrix exponential. It is pointed to several applications of this effective approach in some control tasks.

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