# Application of Hausdorff's Window Function by FIR Filters Synthesis

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*Abstract* - **In this article a new window function, obtained from algebraic polynomial, approximating delta function in Hausdorff metric is presented. Equations defining the polynomial parameters are derived. A domain of definition and analytical equations for Hausdorff's window are defined. The change of Hausdorff's window due to the Hausdorff's distance and the order of the polynomial are graphically presented. Based on the obtained relations, a method for digital FIR filter synthesis is proposed. Equations for impulse and frequencies responses are defined. Mathematical relations for filter order and Hausdorff's distance, due to the attenuation in stop band and the value of transition length, are presented. An example for filter synthesis is shown. A comparative analysis of the magnitude responses of Hausdorff's and Kaiser FIR filters is conducted.**

*Keywords* - **FIR filter, window, Hausdorff, function.**

## I. INTRODUCTION

The window method is one of the ways for FIR filters synthesis. The method's concept is approximation of frequency characteristic of ideal low pass filter.

$$H_d\left(e^{j\omega}\right)=\begin{cases}1e^{j\omega}, & |\omega|\le\omega_c\\ 0 & ,\omega_c<|\omega|\le\pi\end{cases}, \qquad (1)$$

where $\omega_c$ is the cut off frequency of the filter. The impulse response of an ideal filter can be obtained from the equation

$$h_d\left(n\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}H_d\left(e^{j\omega}\right)e^{j\omega n}d\omega . \qquad (2)$$

The ideal impulse response must be limited as to obtain a concrete filter, because the real impulse response must be zero for negative values of its argument $n$ and defined in the interval $0\le k\le N$, where $N$ is the real filter length.

$$h\left(n\right)=\begin{cases}h_d\left(n\right), & 0\le n\le N\\ 0\end{cases}. \qquad (3)$$

The limitation of ideal impulse response could be obtained with multiplying of window function $w\left(n\right)$

$$h\left(n\right)=h_d\left(n\right)w\left(n\right). \qquad (4)$$

In frequency domain, this is equivalent to convolution of ideal filter transfer function with frequency response of window function. This convolution provokes decreasing of steepness between band pass and band stop area and oscillations in the two bands in the proximity of the cut off frequency, due to the spectral deposition effect. Window functions are constant: Bartlett, von Hann, Hamming, Blackman, etc. and variable with parameter: Gauss, Tukey, Chebyshev, Kaiser and so on [1]. From above mentioned windows the best ratio between steepness of magnitude response and attenuation in the stop band gives the Kaiser's window.

In this article a new window function basing on Hausdorff's distance and its application in FIR filers' synthesis is offered.

## II. WINDOW FUNCTION IN HAUSDORFF'S METRIC

Window function is obtained by delta function approximation like this

$$\delta(x)=\begin{cases}[0,M] & x=0\\ 0 & x\ne 0\end{cases} \qquad (5)$$

with an algebraic polynomial, accomplishing the best approximation of delta function in Hausdorff's metric in interval [-1, 1] [2]. On Fig.1 the proximity concerning Hausdorff's distance of the function $M\delta\left(x\right)$ by $M=1$ is shown.

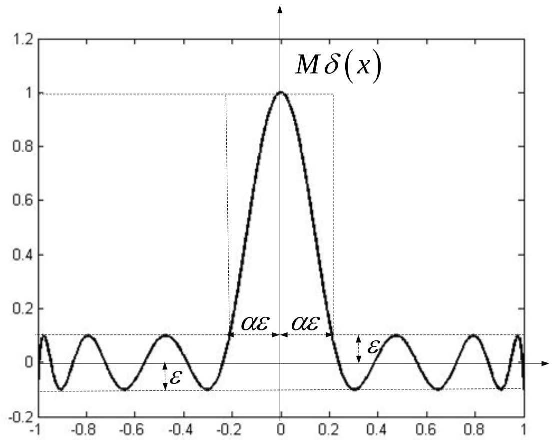


Fig.1. Approximation of delta function with Hausdorff's polynomial

It is proved [2], that the polynomial

$$P_m\left(x\right)=\varepsilon T_m\left(\frac{2x^2-1-\alpha^2\varepsilon^2}{1-\alpha^2\varepsilon^2}\right). \qquad (6)$$

is the unique and the best approximation of delta function in Hausdorff's metric. $T_m$ is Chebyshev's polynomial of first kind and degree *m*; $\alpha$ is parameter, and the factor (product) $\alpha\varepsilon$ determines the function's bandwidth in the area of the $\varepsilon$ level main lobe. The relations between the polynomial's parameters can be defined from the equation

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$$\alpha\varepsilon = \sqrt{\frac{ch\left[\frac{1}{m}Ach\left(\frac{1}{\varepsilon}\right)\right]-1}{ch\left[\frac{1}{m}Ach\left(\frac{1}{\varepsilon}\right)\right]+1}}\,. \qquad (7)$$

The window function is obtained with the translating of Hausdorff's polynomial in positive direction with value 1, the definition domain is reduced only to the main lobe interval $\left[1-\alpha\varepsilon, 1+\alpha\varepsilon\right]$ and raised to the power of 1.27

$$w_m(x) = \left\{\varepsilon T_m\left[\frac{2\left(\alpha\varepsilon x-\alpha\varepsilon\right)^2-1-\alpha^2\varepsilon^2}{1-\alpha^2\varepsilon^2}\right]\right\}^{1.27} =$$

$$= \left\{\varepsilon\cos\left[m\arccos\left|\frac{2\left(\alpha\varepsilon x-\alpha\varepsilon\right)^2-1-\alpha^2\varepsilon^2}{1-\alpha^2\varepsilon^2}\right|\right]\right\}^{1.27}. \qquad (8)$$

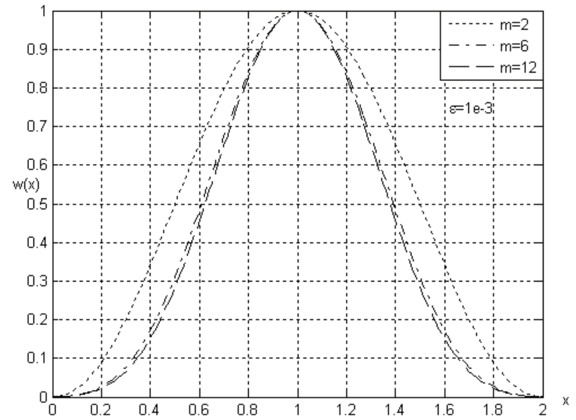


Fig.2. Hausdorff's window functions depending on the power of polynomial $m$

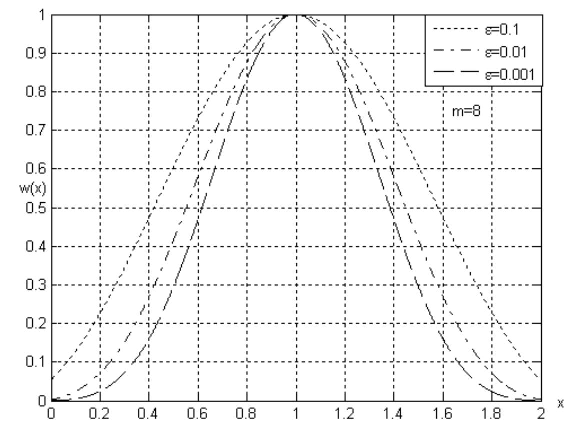


Fig.3. Hausdorff's window functions depending on the Hausdorff's distance $\varepsilon$

On Fig.2 and Fig.3 Hausdorff's window functions are shown, depending on the power of polynomial $m$ values and Hausdorff's distance $\varepsilon$. From the figures a conclusion can be made, that $m$ and $\varepsilon$ change the width of the function main lobe and they can be used as parameters in the FIR filter synthesis.

## III. FIR Filter Synthesis with Hausdorff's Window Function

As it was mentioned in the beginning, the synthesis means an approximation of ideal low pass filter. If the Discrete Fourier Transformation is applied to the ideal magnitude response, an ideal impulse response is obtained in the form of $\sin(x)/x$. To be causal, the function is multiplied by appropriate window function (4) and translated in positive direction with the half of the interval of the window definition domain. The obtained result is the real impulse filter response.

If a sequence of impulses is applied to the digital filter input, then the output response is the sum

$$y(n) = h(1)x(n)+h(2)x(n-1)+$$
$$+h(3)x(n-2)+\cdots+h(N+1)x(n-N), \qquad (9)$$

where $n=[0,N]$.

The equation (9), expressed by the complex variable $z$, is the real filter transfer function and obtains the form

$$H(z) = h(1)z^{-N}+h(2)z^{-N-1}+$$
$$+h(3)z^{-N-2}+\cdots+h(N+1)z^0. \qquad (10)$$

The complex variable $z = re^{j\omega}$ is shown by its module $r$ and angle $\omega$. If we admit $r=1$, then the function $H(z)$ will circumscribe round a single circle the frequency characteristic $H(j\omega)$. Considering this, the transfer function of the real filter is obtained after substitution $z=e^{j\omega}$ in equation (10)

$$H(j\omega) = h(1)e^{-j0\omega}+h(2)e^{-j1\omega}+$$
$$+h(3)e^{-j2\omega}+\cdots+h(N+1)e^{-jN\omega}, \qquad (11)$$

where $h(1), h(2), \dots, h(N+1)$ are the filter coefficients.

To obtain the relations between the filters' parameters is appropriate to use the relation by the Kaiser's

$$N = \operatorname{int}\left(1+\frac{a-7.95}{14.36\Delta f}\right), \qquad (12)$$

where

$$\Delta f = \frac{f_a}{f_s}-\frac{f_c}{f_s} \qquad (13)$$

is the difference between the normalized toward sampling frequency $f_s$ stop band frequency $f_a$ and cut off frequency $f_c$. The attenuation in the stop band in dB is marked by $a$. The Hausdorff's distance could be obtained from the relations:

At $a<24$dB

$$\varepsilon = 1. \qquad (14)$$

At $24\text{dB} \le a \le 50\text{dB}$

$$\varepsilon = \frac{0.66}{\left(2.7\times10^{-5}a^2-8\times10^{-4}a+1.073\right)^{a-25}}. \qquad (15)$$

At $50\text{dB} < a \le 130\text{dB}$

$$\varepsilon = \frac{0.66}{1.1035^{a-25}}. \qquad (16)$$

At $a > 130\text{dB}$

$$\varepsilon = \frac{0.66}{\left(0.0001a + 1.09\right)^{a-25}}. \qquad (17)$$

It is appropriate the power of Hausdorff's polynomial $m$ to be equal to $N$, then from the equation (7) can be defined the factor $\alpha\varepsilon$.

The described method will be shown with the following **example:**

Let us calculate digital FIR filter coefficients with cut-off frequency $f_c = 1\text{Hz}$, stop band frequency $f_a = 2\text{Hz}$, stop band attenuation $a = 25\text{dB}$, at sampling frequency $f_s = 10\text{Hz}$.

From (13) $\Delta f = 0.1$ is defined and from (12)

$$N = \text{int}\left(1 + \frac{25 - 7.95}{14.36 \times 0.1}\right) = 13 = m. \qquad (18)$$

Hausdorff's distance can be obtained from (15)

$$\varepsilon = \frac{0.66}{\left(2.7\times10^{-5} \times 25^2 - 8\times10^{-4} \times 25 + 1.073\right)^{25-25}} = 0.66, \qquad (19)$$

then from (7) the factor $\alpha\varepsilon$ is defined

$$\alpha\varepsilon = \sqrt{\frac{ch\left[\frac{1}{13} Ach\left(\frac{1}{0.66}\right)\right] - 1}{ch\left[\frac{1}{13} Ach\left(\frac{1}{0.66}\right)\right] + 1}} = 0.0375 \qquad (20)$$

The filter's impulse response is obtained from multiplying $\sin(x)/x$ function by Hausdorff's window function (8)

$$h(n+1) = \frac{\sin\left[2\pi \frac{f_c}{f_s}\left(n - \frac{N}{2}\right)\right]}{\pi\left(n - \frac{N}{2}\right)} \times$$

$$\times \left\{ \varepsilon \cos\left[ m \arccos\left| \frac{2\left(\alpha\varepsilon \frac{2n}{N} - \alpha\varepsilon\right)^2 - 1 - \alpha^2\varepsilon^2}{1 - \alpha^2\varepsilon^2} \right| \right] \right\}^{1.27}. \qquad (21)$$

with argument's value $n = 0, 1, 2, \ldots, N$. The obtained values are filter's coefficients by the power of $z^{-n}$

$$h(z) = -0.0234z^{-13} - 0.0124z^{-12} + 0.0173z^{-11} +$$
$$+0.064z^{-10} + 0.1187z^{-9} + 0.1675z^{-8} + 0.1962z^{-7} +$$
$$+0.1962z^{-6} + 0.1675z^{-5} + 0.1187z^{-4} + 0.064z^{-3} +$$
$$+0.0173z^{-2} - 0.0124z^{-1} - 0.0234. \qquad (22)$$

The filter's transfer function is obtained in accordance with (11)

$$H(j\omega) = -0.0234e^{-j0\omega} - 0.0124e^{-j1\omega} +$$
$$+0.0173e^{-j2\omega} + 0.064e^{-j3\omega} + 0.1187e^{-j4\omega} +$$
$$+0.1675e^{-j5\omega} + \cdots - 0.0124e^{-j12\omega} - 0.0234e^{-j13\omega}, \qquad (23)$$

where $\omega = 2\pi f / f_s$; $\left(f = 0 \div f_s/2\right)$.

The transfer function module is the filter magnitude response, and its argument – phase response. In some cases it is necessary the module to be normalised toward 0dB, as dividing by the sum of coefficients $h(n+1)$.

$$H^*(\omega) = \frac{\left|H(j\omega)\right|}{\sum_{n=0}^{N} h(n+1)}. \qquad (24)$$

On Fig.4 and Fig.5 the filter's frequency responses are shown.

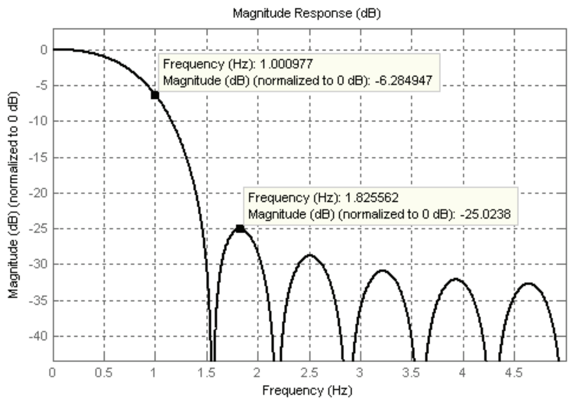


Fig.4. Normalised filter's magnitude response

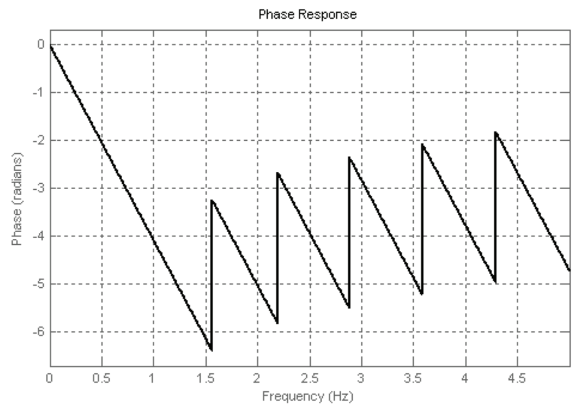


Fig.5. Filter's phase response

From Fig.5 is seen, that the filter has linear phase response. It is because of impulse response symmetry, what comes from the equation (23) coefficients.

The filter's group time delay (GDT) is the phase response derivative. As it is linear, GDT will be constant. Its value is defined from the equation

$$\tau = \frac{N}{2 f_s} = \frac{13}{2 \times 10} = 0.65\text{sec}. \qquad (25)$$

On Fig. 7 magnitude responses of filters with Hausdorff's window and Kaiser's window with equal input data are compared: cut off frequency 1Hz, stop band frequency 2Hz; sampling frequency 10Hz; filter's length $N = 37$ and stop band attenuation 60dB.

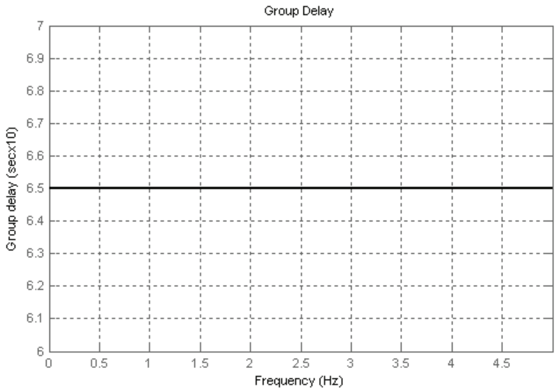


Fig.6. Group Delay Time

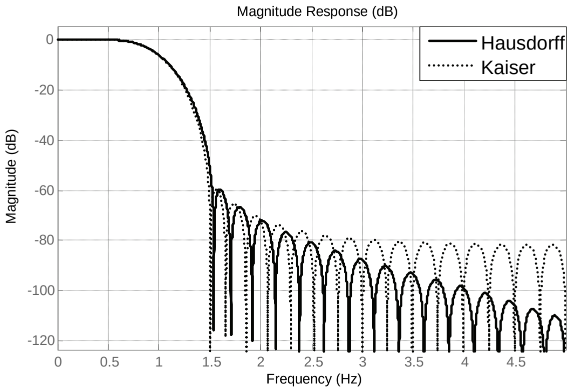


Fig.7. Magnitude responses comparison

The comparison shows that filters are obtained with Hausdorff’s window function, where attenuation in stop band increase quicker than in the Kaiser’s filters. In this case for the frequency band 4-4.5Hz, it is about 18-25dB. This advantage is due to smaller magnitude response steepness in the area between cut off frequency and stop band frequency. In our case for frequency 1.4Hz the difference is about 3.5dB, which is illustrated on Fig.8.

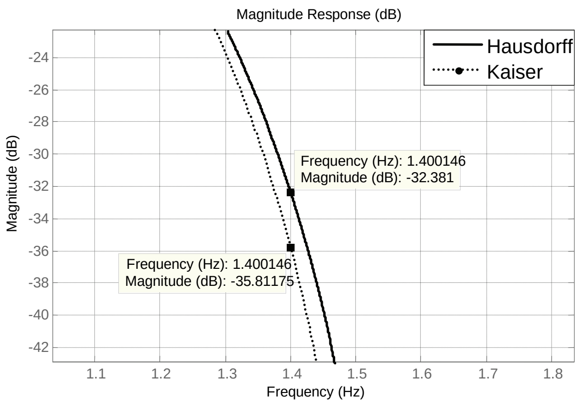


Fig.8. Magnitude responses comparison - fragment

## IV. CONCLUSIONS

In this article a new window function in Hausdorff’s metric is offered. It is applied for the first time in FIR filter synthesis. The obtained characteristics are similar to Kaiser’s filters. The magnitude response has smaller steepness in the area between cut off frequency and stop band frequency. This circumstance defines the bigger attenuation in stop band area. Hausdorff’s FIR filters possess all advantages and disadvantages of this kind of filters. They are calculated easier than digital IIR filters; always are causal; possess linear phase response.

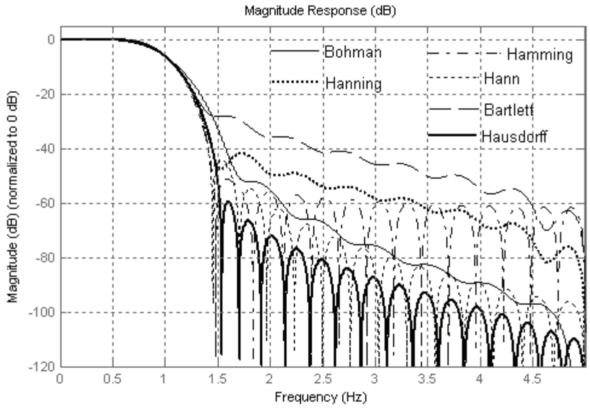


Fig.9. Magnitude responses comparison

The main disadvantages are the smaller selectivity and the impossibility to obtain accurate magnitude response.

The digital FIR filters obtained by Hausdorff’s window function are better than the most FIR filters, as it is illustrated on Fig.9. They expand the variety of filters of this type and may be applied in practice.

## REFERENCES

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[2] Sendov, B. “Hausdorff’s approximations”.*Kluwer Academic Publishers* London 1990, ISBN: 0792309014.