Level Crossing Rate of Phase Process and FM Noise in Nakagami-q Fading Channel Influenced by Interference

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Abstract – **The phase crossing statistics and random FM noise are studied for Nakagami-q fading channel model, in the presense of interference. A closed-form joint PDF of phase process and random FM noise is derived. Expression for the level - crossing rate of the phase process when crossing arbitrary level of phase, is also obtained. All results are compared with the known results in special case of zero interference amplitude. Moreover it is also considered the special case when Nakagami-q fading channel reduces to Rayleigh fading channel. Several plots are given to show how derived values evolve when an amplitude of interference increases.**

Keywords—**Level crossing rate theory, Nakagami-q fading channels, phase process, FM noise, interference.**

I. Introduction

In wireless communications, the signal phase and envelope fluctuates randomly throughout the propagation environment in a fast fading condition. The most representative distributions, used to describe signal anvelope are the Rayleigh, Rice, Hoyt (Nakagami-q), and Nakagami-m. The Nakagami-q fading channel model [2] is one of the often used channel models for the description of the statistics of envelope fading and phase fluctuations in narrowband mobile communication systems.

The Nakagami-q model has been proposed originally as a suitable stochastical model to describe the distribution of the signal amplitude recorded on satellite links subject to ionospheric scintillation [2].

Recently, it has been used more and more frequently in performance analysis of mobile radio communications [3], [4]. Furthermore, it is shown in [5] and [6] that this model is applicable for describing the statistics of the fading envelope of real-world mobile radio channels.

In particular, the Hoyt model [2] considers the in-phase and quadrature signal components as Gaussians with zero means and arbitrary variances. For the case of identical variances, the Hoyt distribution reduces to the Rayleigh. In recent years, different statistics concerning the Hoyt model have been investigated [1,4,9]. In [9], authors derived exact, closed-form, and general expressions of the marginal and joint moments as well as of the correlation coefficient of the instantaneous powers of two Nakagami-q signals.

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The statistical properties of the phase process and its time derivative, known has random frequency modulation (FM) noise, are also of interest in some applications. For example, these properties plays an important role in the design of optimal carrier recovery schemes needed in the synchronization subsystem of coherent receivers [13]. A pioneering work in this matter was carried out by Rice in his classical paper [14], in which the aim was to evaluate the click noise in FM systems, assuming the noise spectrum to be symmetric about the sine wave frequency. Also, in the FM receivers which using a limiter-discriminator for detection random FM spikes generated by phase jumps deteriorate the error-rate performance [15].

In all cases, the level crossing theory plays a central role in the determination of the statistical properties of the channel phase and random FM noise. Level crossing rate (LCR) of the phase process and average fade duration (AFD) are important second-order statistical quantities, extensively explored in the literature. The rapidity of the fading is given by level crossing rate. It quantifies how often the anvelope or phase process crosses some level, usually in the forward direction. The average fade duration is the expected time when the signal envelope (or phase) is continuously below the specified threshold. Closed form of the level crossing rate quantity for Nakagami-q fading channel model without presence of the interference is derived in [1]. But as far as the authors know, no results are reported for channel in the presence of interference with constant amplitude and the same carrier frequency as the useful signal.

The remainder of this paper is organized as follows. In the Section II, we derived closed form for the joint probability density function (JPDF) of phase and random FM noise (phase derivative) when interference is presented. Results are compared with the special case of zero interference amplitude, considered in [1]. Section III discusses phase level - crossing rate which is derived in the closed form. The special cases of this statistics, for specific values of interference amplitude and crossing level, are also considered. In the Section V we plotted obtained PDFs and showed how they evolve when interference amplitude increases.

II. Closed-Form JPDF of the Phase Process and Random FM Noise

The received signal a narrowband Nakagami-q fading channel with presence of interference is described by

$$x_r(t) = X_1(t)\cos\omega t + X_2(t)\sin\omega t + A\cos\omega t\,. \qquad (1)$$

Here $X_1(t)$ and $X_2(t)$ are uncorrelated Gaussian processes with zero mean value and variances σ_1 and σ_2 respectively. Also value A is the interference amplitude. In the equivalent complex baseband, the received signal is:

$$X_r(t) = X_1(t) + A + jX_2(t) \tag{2}$$

In the case $A = 0$ we have no input signal, so situation is the same as in [1]. We will define the amplitude and phase of the process

$$\begin{aligned} R(t) &= \sqrt{(X_1(t)+A)^2 + X_2^2(t)}, \\ \vartheta(t) &= \arctan\left(X_2(t)/(X_1(t)+A)\right) \end{aligned} \tag{3}$$

For $A = 0$, the envelope and phase first-order statistics can be found in [2]. Corresponding statistics can be similarly derived in general case when $A > 0$.

Time derivative of the phase process $\dot{\vartheta}(t)$ is known in the literature (see for example [3]) as random FM noise. To derive PDF of the random FM noise and crossing statistics we will need PDF of the joint process $(X_1, X_2, \dot{X}_1, \dot{X}_2)$. For the symmetrical Doppler power spectral density (PSD), where X_i and $\dot{X}_i$, (*i=1,2*) are in pairs uncorrelated processes [3] it can be shown that joint PDF of $(X_1, X_2, \dot{X}_1, \dot{X}_2)$ is equal to

$$\begin{aligned} p_{X_1X_2\dot{X}_1\dot{X}_2}(x_1, x_2, \dot{x}_1, \dot{x}_2) &= \left(4\pi^2\sigma_1\sigma_2\sqrt{\beta_1\beta_2}\right)^{-1} \times \\ &\times \exp\left(-\frac{1}{2}\left(\frac{x_1^2}{\sigma_1^2} + \frac{x_2^2}{\sigma_2^2} + \frac{\dot{x}_1^2}{\beta_1} + \frac{\dot{x}_2^2}{\beta_2}\right)\right) \end{aligned} \tag{4}$$

where β_i (*i=1,2*) is the variation of the process $\dot{X}_i(t)$. Let we mention that β_i can be expressed as $-\ddot{r}_{X_1X_1}(0)$, i.e. it is the negative curvature of the autocorrelation function $\ddot{r}_{X_1X_1}(\tau)$ at $\tau = 0$ [1,3]. For the classical Jakes Doppler PSD [3], the parameter β_i may be written as $2\left(\pi\sigma_i f_{\max i}\right)^2$, where $f_{\max i}$ denotes the maximum Doppler frequency of the Gaussian process. The PDF of random FM noise is given by:

$$p_{\dot{\vartheta}}(\dot{\theta}) = \int_0^{+\infty}\int_{-\infty}^{+\infty}\int_{-\pi}^{\pi} p_{R\dot{R}\vartheta\dot{\vartheta}}(r, \dot{r}, \theta, \dot{\theta})\, d\theta\, d\dot{r}\, dr \tag{5}$$

Now we can obtain joint PDF of the process $(R, \dot{R}, \vartheta, \dot{\vartheta})$. For this purpose, introduce the transformation of the Cartesian coordinates (x_1, x_2) to shifted polar coordinate system (R, ϑ). Transformation formulae are

$$\begin{aligned} x_1 &= R\cos\vartheta - A, \quad \dot{x}_1 = \dot{R}\cos\vartheta - R\dot{\vartheta}\sin\vartheta \\ x_2 &= R\sin\vartheta, \quad \dot{x}_2 = \dot{R}\sin\vartheta + R\dot{\vartheta}\cos\vartheta \end{aligned} \tag{6}$$

Jacobian of the transformation (6) is equal to $J = -R^2$. Now the joint PDF can be derived as

$$\begin{aligned} p_{R\dot{R}\vartheta\dot{\vartheta}}(r, \dot{r}, \varphi, \dot{\varphi}) &= \left(4\pi^2\sigma_1\sigma_2\sqrt{\beta_1\beta_2}\right)^{-1} r^2 \\ &\times \exp\left(-\frac{1}{2}\left(\frac{\left(-A + r\cos\varphi\right)^2}{\sigma_1^2} + \frac{r^2\sin^2\varphi}{\sigma_2^2} + \right.\right. \\ &\left.\left. + \frac{\left(\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi\right)^2}{\beta_1} + \frac{\left(\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi\right)^2}{\beta_2}\right)\right) \end{aligned} \tag{7}$$

Next we will obtain $p_{R\vartheta\dot{\vartheta}}(r, \varphi, \dot{\varphi})$ by performing an integration with respect to variable $\dot{r}$. Notice that $p_{R\dot{R}\vartheta\dot{\vartheta}}(r, \dot{r}, \varphi, \dot{\varphi})$ can be expressed as function of $\dot{r}$ in the form

$$p_{R\dot{R}\vartheta\dot{\vartheta}}(r, \dot{r}, \varphi, \dot{\varphi}) = M\exp\left(-a\dot{r}^2 + b\dot{r} + c\right), \tag{8}$$

where the *M*, *a*, *b* and *c* are parameters. Expression (8) can be integrated into the closed-form. By performing an integration and after some simplifications we get required PDF

$$p_{R\vartheta\dot{\vartheta}}(r, \varphi, \dot{\varphi}) = \frac{r^2\exp\left(-\frac{a_{\varphi,\dot{\varphi}}r^2}{2} + \frac{Ar\cos\varphi}{\sigma_1^2} - \frac{A^2}{2\sigma_1^2}\right)}{\left(2\pi\right)^{3/2}\sigma_1\sigma_2\sqrt{\beta_\varphi}}, \tag{9}$$

where it is denoted

$$\begin{aligned} \sigma_\varphi^2 &= \sigma_1^2\sin^2\varphi + \sigma_2^2\cos^2\varphi \\ \beta_\varphi &= \beta_1^2\sin^2\varphi + \beta_2^2\cos^2\varphi\,. \\ a_{\varphi,\dot{\varphi}} &= \frac{\sigma_\varphi^2}{\sigma_1^2\sigma_2^2} + \frac{\dot{\varphi}^2}{\beta_\varphi}, \end{aligned} \tag{10}$$

Next step is the integration of $p_{R\vartheta\dot{\vartheta}}(r, \varphi, \dot{\varphi})$ with respect to variable r. Direct integration of (9) yields

$$p_{\vartheta\dot{\vartheta}}(\varphi, \dot{\varphi}) = \int_0^{+\infty} p_{R\vartheta\dot{\vartheta}}(r, \varphi, \dot{\varphi})\,dr = \frac{\exp\left(-\frac{A^2}{2\sigma_1^2}\right) f\left(\frac{A\cos\varphi}{\sigma_1^2\sqrt{2a_{\varphi,\dot{\varphi}}}}\right)}{4(\pi a_{\varphi,\dot{\varphi}})^{3/2}\sigma_1\sigma_2\beta_\varphi}. \tag{11}$$

By $f(s)$, the following function is denoted

$$f(s) = 2s + \exp\left(s^2\right)\sqrt{\pi}\left(1 + 2s^2\right)\left(1 + \operatorname{erf}(s)\right), \tag{12}$$

where an error function $\operatorname{erf}(x)$ is defined as usual

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp\left(-t^2\right)dt\,. \tag{13}$$

Note that by taking $A = 0$, (11) reduces to JPDF of phase and FM noise of Nakagami-q signal [1, (4)]

$$\left. p_{\vartheta\dot{\vartheta}}(\varphi, \dot{\varphi})\right|_{A=0} = \frac{a_{\varphi\dot{\varphi}}^{-3/2}}{4\pi^2\sigma_1\sigma_2\beta_\varphi}\,. \tag{14}$$

Now required PDF $p_{\dot{\vartheta}}(\dot{\varphi})$ of random FM noise can be expressed by

$$p_{\dot{\vartheta}}(\dot{\varphi}) = \int_{-\pi}^{\pi} p_{\vartheta\dot{\vartheta}}(\varphi, \dot{\varphi})\,d\varphi\,. \tag{15}$$

Unfortunatelly, last integral cannot be solved into the closed-form. Note that $p_{\dot{\vartheta}}(\dot{\varphi})$ is an even function. Moreover (11) yields that $p_{\vartheta\dot{\vartheta}}(\varphi, \dot{\varphi})$ is even with respect to $\dot{\varphi}$ for every value of φ. Therefore we can conclude that mean value $E\dot{\vartheta}(t)$ of process $\dot{\vartheta}(t)$ is zero. It agrees with the special case from [1] when $A = 0$.

Now consider variance of the random FM noise $\dot{\vartheta}(t)$

$$\sigma_{\dot{\vartheta}}^2 = \int_{-\infty}^{+\infty}\dot{\varphi}^2 p_{\dot{\vartheta}}(\dot{\varphi})\,d\dot{\varphi} = \int_0^{2\pi}\left(\int_{-\infty}^{+\infty}\dot{\varphi}^2 p_{\vartheta\dot{\vartheta}}(\varphi, \dot{\varphi})\,d\dot{\varphi}\right)d\varphi\,. \tag{16}$$

It can be proven that value of the following integral

$$\int_{-\infty}^{+\infty} \dot{\varphi}^2 p_{\vartheta\dot{\vartheta}}(\varphi,\dot{\varphi})d\dot{\varphi} \tag{17}$$

is infinity for each value of φ. This yields immediatelly that $\sigma_{\dot{\vartheta}}^2$ is infinite. Rice [14] considered the quantity $E\left|\dot{\vartheta}(t)\right|$ to obtain the measure of spread of the PDF $p_{\dot{\vartheta}}(\dot{\varphi})$ in the case $\beta_1 = \beta_2 = \beta$, $\sigma_1 = \sigma_2 = \sigma$ and $A = 0$. In the next section we will show that this quantity corresponds to the level - crossing rate of the phase process. More precisely, we will show that $E\left|\dot{\vartheta}(t)\right|$ is an averaged value of expected crossing rate, averaged over all crossing levels.

III. Statistics of the Phase Process Crossing Rate

The level-crossing rate of the phase process $\vartheta(t)$, denoted by $N_\vartheta^+(\varphi_0)$, is the expected number of times the phase process goes through the specified phase level φ_0 with a positive slope. General expression for $N_\vartheta^+(\varphi_0)$, as obtained in [3] is

$$N_\vartheta^+(\varphi) = \int_0^{+\infty} \dot{\varphi} p_{\vartheta\dot{\vartheta}}(\varphi,\dot{\varphi})d\dot{\varphi}. \tag{18}$$

We have already derived joint PDF of phase and FM noise and it is given by (11). In order to find an integral in (18) we will introduce the substitution $t = \frac{A\cos\varphi}{\sigma_1^2\sqrt{2a_{\varphi,\dot{\varphi}}}}$. Applying this substitution to (18) and using

$$\int f(s)ds = \sqrt{\pi}s\left(1+\mathrm{erf(s)}\right)\exp\left(s^2\right), \tag{19}$$

we finally obtain the expression for $N_\vartheta^+(\varphi)$

$$N_\vartheta^+(\varphi) = \frac{\sqrt{\beta_\varphi}}{4\pi\sigma_\varphi}\exp\left(-\frac{A^2\sin^2\varphi}{2\sigma_\varphi^2}\right)\left[1+\mathrm{erf}\left(\frac{A\sigma_2\cos\varphi}{\sigma_1\sigma_\varphi\sqrt{2}}\right)\right]. \tag{20}$$

By taking $A = 0$, expression (20) reduces to expression (10) in [1]

$$N_\vartheta^+(\varphi)\Big|_{A=0} = \frac{\sqrt{\beta_\varphi}}{4\pi\sigma_\varphi}. \tag{21}$$

If we additionally suppose $\beta_1 = \beta_2 = \beta$ and $\sigma_1 = \sigma_2 = \sigma$, then $N_\vartheta^+(\varphi)$ is constant and equal to the $\gamma/4\pi$ where $\gamma = \sqrt{\beta}/\sigma$ is radius of gyration of Doppler PSD of processes $X_1(t)$ and $X_2(t)$. In case when $\beta_1 = \beta_2 = \beta$ and $\sigma_1 = \sigma_2 = \sigma$ but $A \neq 0$, $N_\vartheta^+(\varphi)$ is not constant. In general case, $N_\vartheta^+(\varphi)$ decreases rapidly when A increases. Only for $\varphi = 0$, the exponential part is zero, so we can conclude that $N_\vartheta^+(0)$ increases with A and has the limit value

$$\lim_{A\to+\infty} N_\vartheta^+(0) = \frac{1}{2\pi}\frac{\sqrt{\beta_2}}{\sigma_2} = \frac{\gamma_2}{2\pi} \tag{22}$$

We will also consider the case $\varphi = \pm\pi$. Now also exponential factor is zero, but $N_\vartheta^+(\pm\pi)$ is decreasing due to the factor $1+\mathrm{erf}\left(-\frac{A}{\sigma_1\sqrt{2}}\right)$.

V. Numerical Examples

The PDF of random FM noise and phase level - crossing rate are evaluated at for concrete values of parameters

$$\begin{gathered}\sigma_1^2 = 0.10391, \quad \sigma_2^2 = 0.030488 \\ \beta_1 = 1103.4298\ \mathrm{s}^{-1}, \quad \beta_2 = 1091.5206\ \mathrm{s}^{-1}\end{gathered}. \tag{23}$$

These values are the same as in [1,6] and are obtained by fitting the first and second order statistics of the envelope $R(t)$ of Nakagami-q model to measurement data of an equivalent mobile satellite channel for heavy shadowing enviroment. On fig. 1, phase level - crossing rate $N_\vartheta^+(\varphi)$ is shown for different values of amplitude A. Relation (18) yields that $N_\vartheta^+(\varphi)$ depends on φ only as the function of $\sin^2\varphi$ and $\cos\varphi$. Therefore $N_\vartheta^+(\varphi)$ is an even function on φ and periodic with the period 2π. A 3D plot of $N_\vartheta^+(\varphi)$ as function of φ and A is given on the fig. 2.

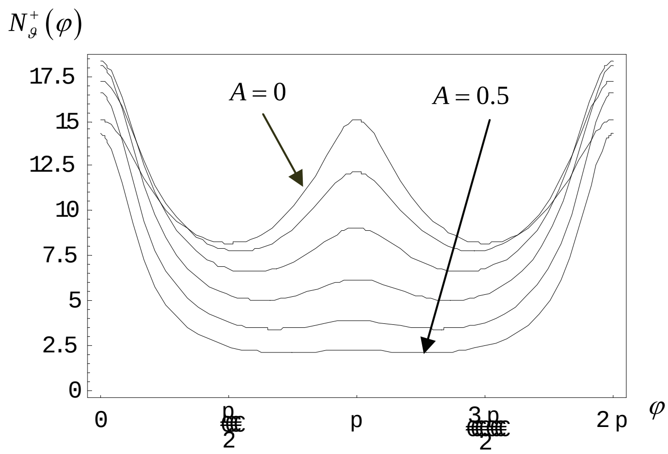

Fig. 1. Average phase crossing rate $N_\vartheta^+(\varphi)$ as a function of variable φ for different values of the amplitude A

Fig. 3 shows 3D plot of the function $p_{\vartheta\dot{\vartheta}}(\varphi,\dot{\varphi})$ for $A = 0.1$. It can be noticed that probability decreases rapidly when $\dot{\varphi}$ increases, which is natural.

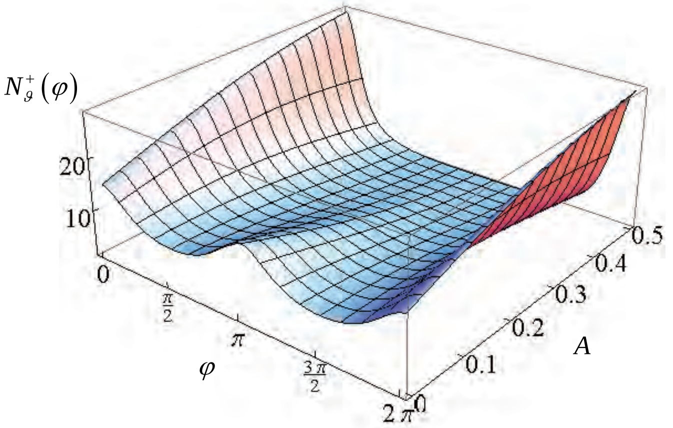

Fig. 2. 3D plot of $N_{\vartheta}^{+}(\varphi)$ as the function of φ and A

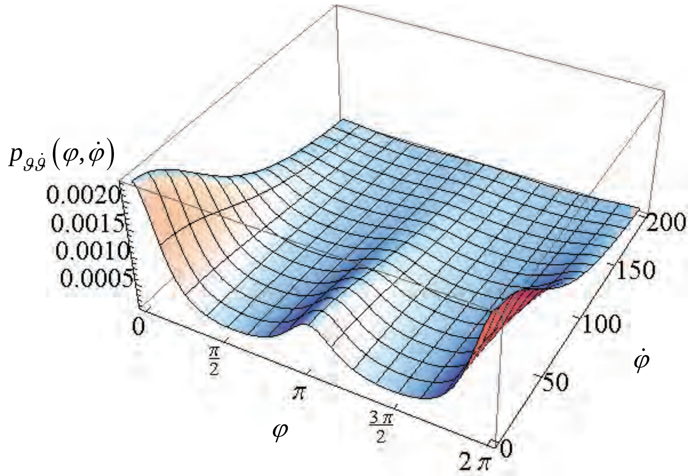

Fig. 3. 3D plot of $p_{\vartheta\dot{\vartheta}}(\varphi,\dot{\varphi})$ for $A = 0.1$

VI. Conclusion

In this paper, we studied the crossing statistics of phase processes and random FM noise encountered in Nakagami-q fading channels with presence of the interference. This results are the extensions of the corresponding results from [1].

We derived the JPDF of the phase process and random FM noise. Also closed-form expressions for the phase level-crossing rate are established.

This theoretical results are useful in for analyzing the statistics of FM spikes during the transmission of the signal over Nakagami-q fading channels with the presence of interference.

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