Designing a Compound Chaotic System Based on the Hide's Model

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Abstract – **In this paper an approach to design a high-order chaotic system with complex dynamics on the base of the well-known Hide model is proposed. The resulted compound chaotic system is obtained using the principles of chaotic synchronization. A synchronization scheme between two compound systems, synthesized by the principle of linear-nonlinear decomposition is also proposed.**

Keywords **– Chaotic systems, Chaotic synchronization, Linear-nonlinear decomposition.**

I. Introduction

During the last two decades a tremendous increase of interest in one specific field of the nonlinear science - the chaotic dynamics, is observed. This is mainly due to the fact that these systems have some properties, which are common as for the stochastic systems as well as for the systems with regular behaviour. It was found that the chaotic systems, due to their intrinsic features, such as a strange attractor in the phase space and a positive Lyapunov exponent, can be used for data protection in secure communication systems or for encrypting text or images. Such systems are based on a phenomenon, called chaotic synchronization, where two or more chaotic systems tune their dynamics to each other.

Most of the known models of chaotic systems are of low-order – mostly third-order continuous chaotic models and two- or third-order discrete chaotic models are known so far. Few fourth-order continuous models and very few fifth- and high-order chaotic models are known. At the same time, it was found that the high-order chaotic systems usually possess more complex dynamics, compared with the low-order models, which can be a significant advantage in the chaotic data protection systems. Using such systems, a higher degree of data security can be achieved.

In this paper a simple yet reliable approach for designing high-order chaotic models is proposed. It is based on the partial replacement chaotic synchronization method. The well-known Hide third-order chaotic model is used to build a sixth-order compound chaotic system. As the main potential application of such system is in data protection systems, some synchronization schemes between two compound Hide systems are proposed. The standard linear-nonlinear decomposition synchronization approach and a new modification of it are used for the schemes. This approach has one significant advantage over the other known synchronization methods – it allows precise stability analysis.

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II. Building Compound Chaotic Systems Using Chaotic Synchronization Principle

Every continuous chaotic system can be presented in the form:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t), \quad (1)$$

where $\mathbf{x} \in \Re^n$ and $\mathbf{f}(\mathbf{x}, t)$ is a nonlinear function of the state variables.

The chaotic synchronization problem can be formulated in the following way: given two (or more) identical chaotic systems of type (1), one has to find a proper coupling between them, such that the two systems evolve identically when they are started from different initial conditions [1]. In the case of uni-directional coupling the system providing the coupling is called *Master system*, and the other system – *Slave system*.

Different synchronization methods exist depending on the type of the coupling. By the popular partial replacement method [5] if the Master system is described by Eq. (1), the Slave system is defined with:

$$\dot{\tilde{\mathbf{x}}} = \mathbf{f}(\tilde{\mathbf{x}}, x_i, t), \quad (2)$$

where $\tilde{\mathbf{x}} \in \Re^n$ is the state vector of the Slave system, $\mathbf{f}(\tilde{\mathbf{x}}, \tilde{x}_i, t) = \mathbf{f}(\mathbf{x}, t)$ and x_i is a state variable from the Master system which substitutes the corresponding variable $\tilde{x}_i$ only on one position in the Slave system's model.

Apparently many possible substitutions in the form of Eq. (2) exist for a given pair of chaotic models. Generally by the synchronization problems one has to find such coupling that:

$$\lim_{t\to\infty} \mathbf{e}(t) = 0, \quad (3)$$

where:

$$\mathbf{e}(t) = \mathbf{x}(t) - \tilde{\mathbf{x}}(t) \quad (4)$$

is the error function between the two state vectors.

If, on the contrary, the coupling is chosen in such way that:

$$\lim_{t\to\infty} \mathbf{e}(t) = m(t), \quad (5)$$

where $m(t)$ is chaotic function, the two chaotic systems (1) and (2) can be assumed as one compound chaotic system of $2n$-th order.

In this case it is obvious, that due to the coupling the dynamics of the Slave subsystem of the compound system (1)-(2) will be subjugated to the Master subsystem and the two subsystems can be viewed as a high-order chaotic generator.

This concept can be generalized for cases, when the parameters of Eqs. (1) and (2) are not equal, as is common with the basic synchronization problems. Then the Master and the Slave subsystems are:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{p}, t), \tag{6}$$

$$\dot{\tilde{\mathbf{x}}} = \mathbf{f}(\tilde{\mathbf{x}}, \tilde{\mathbf{p}}, x_i, t), \tag{7}$$

where $\mathbf{p} \neq \tilde{\mathbf{p}}$ are the parameter vectors of the two systems.

It is even possible to apply this principle to two completely different chaotic systems:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{p}, t), \tag{8}$$

$$\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}, \mathbf{q}, x_i, t), \tag{9}$$

where $\mathbf{y} \in \Re^m$ is the state vector and $\mathbf{q}$ is the parameter vector of the Slave subsystem.

The presented technique allows to artificially obtain high-order software chaotic generators with complex dynamics when such is needed in particular applications. Using this principle, one can also couple more than two simple chaotic systems and obtain a compound system of even higher order.

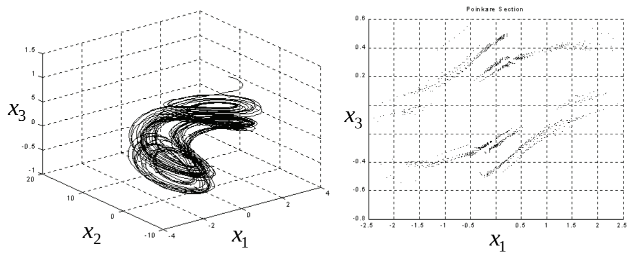
III. Hide Chaotic System

The Hide's model describes an electro-mechanical system with chaotic behaviour [4]. Since the model will be used only as an abstract chaos generator in this paper, the exact system which it describes will not be discussed here. The model's equations are:

$$\begin{aligned} \dot{x}_1 &= x_1 x_2 - x_1 - \beta x_3, \\ \dot{x}_2 &= \alpha(1 - x_1^2) - k x_2, \\ \dot{x}_3 &= x_1 - \lambda x_3, \end{aligned} \tag{10}$$

where the nominal values of the system parameters, for which the system exhibits chaotic behaviour, are: $\beta = 2$, $\alpha = 20$, $\lambda = 1.2$ and $k = 1$.

The system's typical chaotic attractor, built up from a virtually infinite number of unstable periodic orbits, confined in some basin in the phase space, is shown on Fig. 1a. A Poincare section in the plane (x_1, x_3) for $x_2 = 6$ is shown on Fig. 1b. The form of the Poincare section confirms the chaotic nature of the system.



a. b.

Fig. 1. Presence of chaos in the Hide system. a - chaotic attractor, b – Poincare section

IV. Compound 6-th Order Chaotic System on the Basis of Hide's System

A synchronization scheme of the type of Eqs. (6) and (7) is built for the Hide's model. The Master subsystem is described by Eq. (10). After some research, a coupling that satisfies Eq. (5) is found. The coupling variable is x_2 in the second equation of the Slave subsystem:

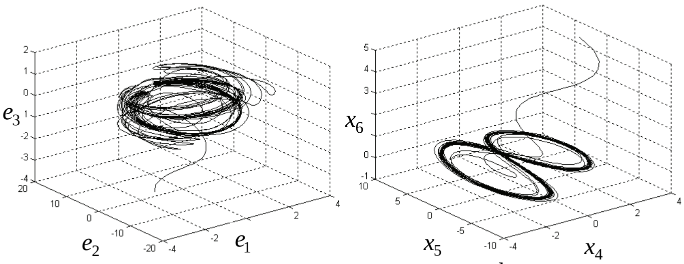
$$\begin{aligned} \dot{\tilde{x}}_1 &= \tilde{x}_1 \tilde{x}_2 - \tilde{x}_1 - \tilde{\beta} \tilde{x}_3, \\ \dot{\tilde{x}}_2 &= \tilde{\alpha}(1 - \tilde{x}_1^2) - \tilde{k}\, \underline{\underline{x_2}}, \\ \dot{\tilde{x}}_3 &= \tilde{x}_1 - \tilde{\lambda} \tilde{x}_3. \end{aligned} \tag{11}$$

The parameters of Eq. (11) are chosen to be different from those of Eq. (10): $\tilde{\beta} = 2.2$, $\tilde{\alpha} = 22$, $\tilde{\lambda} = 1.3$ and $\tilde{k} = 1.1$.

For convenience the compound system (10)-(11) can be rewritten assuming that $\tilde{\mathbf{x}} = [\tilde{x}_1 \quad \tilde{x}_2 \quad \tilde{x}_3]^T = [x_4 \quad x_5 \quad x_6]^T$:

$$\begin{aligned} \dot{x}_1 &= x_1 x_2 - x_1 - \beta x_3, \\ \dot{x}_2 &= \alpha(1 - x_1^2) - k x_2, \\ \dot{x}_3 &= x_1 - \lambda x_3, \\ \dot{x}_4 &= x_4 x_5 - x_4 - \tilde{\beta} x_6, \\ \dot{x}_5 &= \tilde{\alpha}(1 - x_4^2) - \tilde{k} x_2, \\ \dot{x}_6 &= x_4 - \tilde{\lambda} x_6. \end{aligned} \tag{12}$$

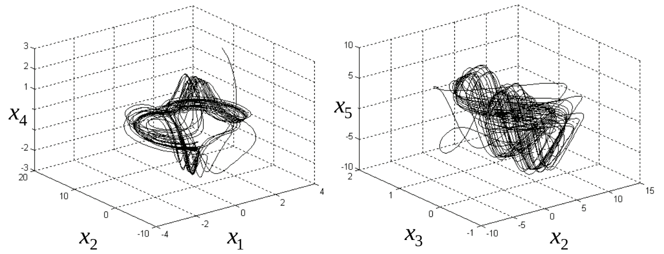
It was proved by simulation with Simulink that for the chosen coupling Eq. (5) holds. The state space of the error function (e_1, e_2, e_3), $e_i = x_i - \tilde{x}_i$, is shown on Fig. 2a. It is evident that the error system attractor is chaotic. The attractor of the Slave subsystem is shown on Fig. 2b. Apparently, this attractor is of completely different shape, compared to the attractor of the Master subsystem, shown on Fig. 1a. Thus, by the simple coupling with the x_2 variable the dynamics of the Slave subsystem becomes conjugated to that of the Master system, it is chaotic too, but different from the basic Hide attractor and therefore the compound system, described by Eq. (12) can be considered as a 6-th order chaotic generator.



a. b.

Fig. 2. Compound Hide system. a - attractor of the error function, b – attractor of the Slave subsystem

To confirm the complex dependence between the variables of the Master and the Slave subsystems of the Hide's compound system, two of the „mixed" state subspaces, formed by variables of $\mathbf{x}$ and $\tilde{\mathbf{x}}$, are shown on Fig. 3. The attractors in the subspaces (x_1, x_2, x_4) and (x_2, x_3, x_5) are apparently chaotic and of different shape and size, compared to the basic Hide attractor.



a. b.

Fig. 3. Attractor of the compound Hide system.
a - (x_1, x_2, x_4) subspace, b - (x_2, x_3, x_5) subspace

V. SYNCHRONIZATION BETWEEN TWO COMPOUND HIDE SYSTEMS

Given the compound Hide system, it is important to find some stable synchronization schemes between **two** such **identical** systems with a view to the possible application of this system in data protection systems.

Two different synchronization schemes on the basis of the **linear-nonlinear decomposition** method are presented below. This synchronization approach [7] is recommended for cases, when the precise stability analysis of the synchronization manifold is crucial. The essence of the method is in the following: *given a continuous chaotic system of type* (1), *one can separate the linear and the nonlinear parts of the system, if this is possible, in the following way*:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{h}(\mathbf{x}(t), t), \tag{13}$$

where $\mathbf{h}(\mathbf{x}(t), t)$ contains ***all*** *nonlinear terms* of Eq. (1) and the $\mathbf{A}$ matrix contains the *linear terms*.

If Eq. (13) is assumed to be a Master system, the Slave system is *built as a copy of the linear part of* (13), *driven by the nonlinear terms*:

$$\dot{\tilde{\mathbf{x}}}(t) = \mathbf{A}\tilde{\mathbf{x}}(t) + \mathbf{h}(\mathbf{x}(t), t). \tag{14}$$

The error system, obtained by subtracting (14) from (13), is then a *linear system*:

$$\dot{\mathbf{e}}(t) = \mathbf{A}(\mathbf{x}(t) - \tilde{\mathbf{x}}(t)) = \mathbf{A}\mathbf{e}(t). \tag{15}$$

Thus, the stability of the synchronization manifold is proven simply by *calculating the eigenvalues of the* $\mathbf{A}$ *matrix*.

Given the compound Hide system with Eq. (12), it can easily be decomposed in the form of Eq. (13) with linear part:

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & -\beta & 0 & 0 & 0 \\ 0 & -k & 0 & 0 & 0 & 0 \\ 1 & 0 & -\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -\tilde{\beta} \\ 0 & -\tilde{k} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -\tilde{\lambda} \end{bmatrix}, \tag{16}$$

and a nonlinear part:

$$\mathbf{h}(\mathbf{x}(t), t) = \begin{bmatrix} x_1 x_2 \\ \alpha(1 - x_1^2) \\ 0 \\ x_4 x_5 \\ \tilde{\alpha}(1 - x_4^2) \\ 0 \end{bmatrix}. \tag{17}$$

Then, if Eq. (12) is the Master system of the synchronization scheme, the Slave system is:

$$\begin{aligned} \dot{\tilde{x}}_1 &= \underline{x_1 x_2} - \tilde{x}_1 - \beta \tilde{x}_3, \\ \dot{\tilde{x}}_2 &= \alpha(1 - \underline{x_1^2}) - k\tilde{x}_2, \\ \dot{\tilde{x}}_3 &= \tilde{x}_1 - \lambda \tilde{x}_3, \\ \dot{\tilde{x}}_4 &= \underline{x_4 x_5} - \tilde{x}_4 - \tilde{\beta} \tilde{x}_6, \\ \dot{\tilde{x}}_5 &= \tilde{\alpha}(1 - \underline{x_4^2}) - \tilde{k}\tilde{x}_2, \\ \dot{\tilde{x}}_6 &= \tilde{x}_4 - \tilde{\lambda}\tilde{x}_6. \end{aligned} \tag{18}$$

The stability of the synchronization scheme, defined by Eqs. (12),(18) can be proved by calculating the eigenvalues of Eq. (16), which for the given set of parameters are:

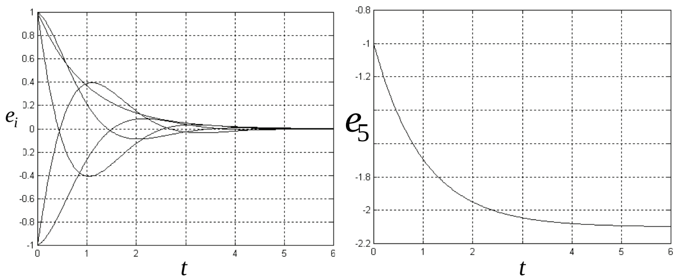
$$\rho_1 = 0, \ \rho_2 = -1, \ \rho_{3,4} = -1.1 \pm 1.4j, \ \rho_{5,6} = -1.1 \pm 1.5j. \tag{19}$$

A synchronization scheme is stable and Eq. (3) is fulfilled when *all* eigenvalues of the $\mathbf{A}$ matrix are with negative real parts. However, if the maximum eigenvalue is zero, as is the case here, a more complex type of synchronization between the systems (12) and (18) occurs. It is called *marginal synchronization* [3,6] and is characterized by:

$$\lim_{t \to \infty} \mathbf{e}(t) = c, \tag{20}$$

where c is a constant, depending on the initial conditions of the two systems.

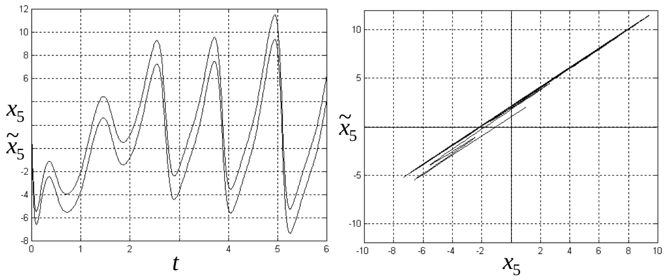
The synchronization scheme is simulated with Simulink and the presence of marginal synchronization was confirmed. The only variable of the error vector **e**, different of zero, is $e_5 = x_5 - \tilde{x}_5$. For the chosen set of initial conditions - $\mathbf{x}(0) = [2 \ \ 3 \ \ 1 \ \ 3 \ \ 1 \ \ 5]^T$ and $\tilde{\mathbf{x}}(0) = [3 \ \ 2 \ \ 2 \ \ 2 \ \ 2 \ \ 4]^T$, the non-zero error is $e_5 = 2.1$. It was found that only the initial conditions $\mathbf{x}(5)$ and $\tilde{\mathbf{x}}(5)$ affect the value of e_5. For example changing $\tilde{\mathbf{x}}(5)$ to 5 gives marginal synchronization with $e_5 = 5.1$. The error dynamics is shown on Fig. 4.



a. b.

Fig. 4. Error dynamics. a - $e_1(t), e_2(t), e_3(t), e_4(t), e_6(t)$, b - $e_5(t)$

The time series of x_5 and $\tilde{x}_5$ are shown on Fig. 5a. Here the interesting phenomenon of two chaotic signals evolving equally, but with constant separation from each other, can be seen. Fig. 5b shows the projection of the combined attractor of the two compound Hide systems in the phase plane $(x_5, \tilde{x}_5)$.



a. b.

Fig. 5. Marginal synchronization. a - $x_5(t), \tilde{x}_5(t)$,

b – phase plane $(x_5, \tilde{x}_5)$

In order to search for different types of synchronization, namely identical synchronization of type of Eq. (3), the linear-nonlinear decomposition method can be further extended by introducing a linear feedback coupling into the Slave system:

$$\dot{\tilde{\mathbf{x}}}(t) = \mathbf{A}\tilde{\mathbf{x}}(t) + \mathbf{h}(\mathbf{x}(t),t) + \boldsymbol{\alpha}_{fb}\mathbf{E}(\mathbf{x}(t) - \tilde{\mathbf{x}}(t)), \qquad (21)$$

where $\boldsymbol{\alpha}_{fb}$ is the coupling gain vector and $\mathbf{E}$ is the coupling matrix.

Thus, the error system remains linear:

$$\dot{\mathbf{e}}(t) = (\mathbf{A} - \boldsymbol{\alpha}_{fb}\mathbf{E})\mathbf{e}(t), \qquad (22)$$

but now there exist many possibilities to tune the coupling by introducing different types of feedback.

If in the synchronization scheme (12),(18) a feedback coupling with $\boldsymbol{\alpha}_{fb} = [0 \;\; 0 \;\; 0 \;\; 0 \;\; 10 \;\; 0]$ is introduced, i.e. the fifth equation of the Slave system is changed with:

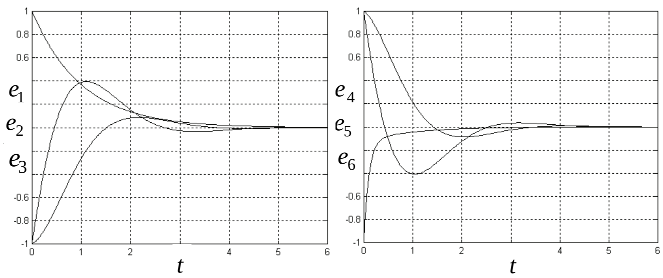
$$\dot{\tilde{x}}_5 = \tilde{\alpha}(1 - \underline{x_4^2}) - \tilde{k}\tilde{x}_2 + \underline{10(x_5 - \tilde{x}_5)}, \qquad (23)$$

the eigenvalues of the linear $(\mathbf{A} - \boldsymbol{\alpha}_{fb}\mathbf{E})$ matrix are:

$$\lambda_1 = -1,\ \lambda_2 = -10,\ \lambda_{3,4} = -1.1 \pm 1.4j,\ \lambda_{5,6} = -1.1 \pm 1.5j, \qquad (24)$$

and because all of them are with negative real parts, the synchronization scheme is stable and identical synchronization will occur.

The simulation results for the modified feedback coupling with Eq. (23) are presented on Fig. 6.



a. b.

Fig. 6. Identical synchronization. a - $e_1(t) \div e_3(t)$,b - $e_4(t) \div e_6(t)$

On Fig. 7 the time evolution of x_5 and $\tilde{x}_5$, and the phase plane $(x_5, \tilde{x}_5)$ are shown. Apparently the two chaotic signals evolve identically, such is the case for all other pairs $x_i, \tilde{x}_i$.

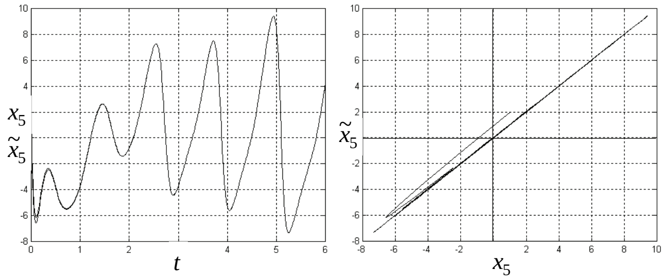


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Fig. 7. Identical synchronization. a - $x_5(t), \tilde{x}_5(t)$,

b – phase plane $(x_5, \tilde{x}_5)$

V. Conclusion

The proposed approach to build high-order compound chaotic systems provides a simple yet efficient tool to generate complex chaotic dynamics. Such kind of ostensibly random signal can be used to hide the information signal in different kind of data protection systems – communication systems over the Internet or by radio waves, systems for encrypting text and images etc., which were proposed so far in the real world [2]. In the core of such systems there is always a stable chaotic synchronization scheme.

The applicability of the compound Hide system to build synchronization schemes was shown with two schemes, designed using the linear-nonlinear synchronization approach. The marginal synchronization, achieved by the basic linear-nonlinear coupling, is dependent on the initial conditions, and this can be used to achieve a higher security level in data protection systems.

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