Synchronization of Hyperchaotic Systems with Impermanent One-Variable Coupling

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Abstract – **In this paper synchronization schemes for the Hoff fourth-order hyperchaotic system are proposed. The schemes are designed in such way, that the most economical possible coupling between the master and the slave systems is applied. This is achieved firstly, by finding a fast synchronization scheme with only one-variable coupling, and secondly – by applying the coupling not constantly, but only for short periods of time.**

Keywords – **Chaos, Chaotic synchronization, Feedback coupling**

I. Introduction

Recently there has been growing interest in the investigation of the chaotic synchronization phenomenon. Being nonlinear systems with very special type of dynamic behaviour, more common to the stochastic systems, the chaotic systems were long considered as non-usable in the real-world. Moreover, the chaos was counted as a harmful and non-controllable system state. After 1990, control methods for chaotic systems were invented with the aim to stabilize a given chaotic system in a fixed-point or in a periodic orbit. Then, it was found that chaos can also be useful, and even methods for artificially generating chaos in a preliminary non-chaotic nonlinear systems were proposed. Such is the case in some processes of chaotic mixing in chemical reactors, where the efficiency is increased when the species are fed chaotically in the reactor.

Another field of interest is the implementation of chaotic systems in secure communication systems, where the pseudo-randomness of the chaotic signal is used to mask the information signal. Such systems are based on a very interesting phenomenon, observed by chaotic systems - the chaotic synchronization. It was found that two or more chaotic systems can synchronize their dynamics and evolve identically and at the same time pseudo-randomly, when a suitable coupling is applied between them. Recently, the designing of the synchronization coupling was evolved in a major task in the nonlinear science. Many different kinds of chaotic synchronization methods are proposed so far, each with its advantages and drawbacks, but what is common, is that no universal chaotic synchronization method exists, which can always guarantee the synchronization between a given pair of chaotic systems. This fact predetermines the constant research in this field and frequently new synchronization methods and new modifications of the existing ones are proposed. Every synchronization method can be suitable in some cases and non-usable in others. Some methods aim at increasing the speed of the synchronization, others – at the simplicity of the synchronization scheme, third – at the strong analytical proof of synchronization stability.

In this paper synchronization schemes that satisfy the possible requirement for a most-economical type of coupling between the systems, subjected to synchronization, are proposed. This is achieved by two main approaches. First, a stable synchronization scheme with only one master-system variable, used for the coupling, is designed. To achieve synchronization, a method called combined synchronization approach, is used. This method was proposed by the author earlier and has the advantage of offering many possible couplings between the systems, of which one can choose the most appropriate, in particular the fastest one-variable coupling is chosen for the given case. The second technique, applied in terms of the economy, is based on the principle, that no continuous coupling is necessary to achieve synchronization for most of the known chaotic systems. So, the coupling is switched on only for short periods of time, chosen carefully, and between them the two systems evolve independently.

The experiments are conducted with computer simulations, using the fourth-order Hoff hyperchaotic system as a basis for the synchronization schemes. However, the proposed technique is not confined only to this system and can be used with most of the known continuous chaotic models after some research on the particular system's properties.

II. Principles of Chaotic Synchronization

The synchronization of chaotic systems is a phenomenon, by which two or more such systems synchronize their dynamics in some way. Although synchronization between two completely different chaotic systems is possible (generalized synchronization), one usually deals with the so called identical synchronization – given two identical chaotic systems, being started from different initial conditions, one has to find a coupling between them, such that their dynamics become synchronized and they evolve identically in time. In general, in the case of a one-way coupling, which is more common, the two chaotic systems (called master and slave) can be presented in the form [1]:

$$\text{Master} \qquad \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t), \tag{1}$$

$$\text{Slave} \qquad \dot{\tilde{\mathbf{x}}} = \mathbf{f}(\tilde{\mathbf{x}}, \mathbf{x}, t), \tag{2}$$

where $\mathbf{x} \in \Re^n$, $\tilde{\mathbf{x}} \in \Re^n$ are the state vectors of the two systems.

A basic scheme of the synchronization principle is shown on Fig. 1.

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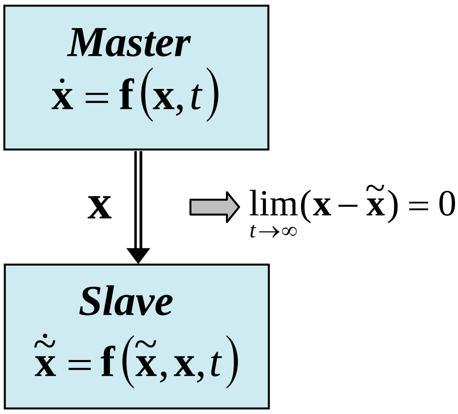

Fig. 1. Basic synchronization principle

The coupling between the Master and the Slave systems has to be designed in such way that:

$$\lim_{t\to\infty} \mathbf{e}(t) = 0\,, \qquad (3)$$

where $\mathbf{e}(t) = \mathbf{x}(t) - \tilde{\mathbf{x}}(t)$ is the “error” between the states of the two systems.

It was found, that continuous coupling is not always needed to achieve synchronization. Some schemes with occasional coupling were proposed [2,5,6]. The common with them is the principle, that two chaotic systems can synchronize if the coupling signal is active only for short periods of time. The slave system is then defined by:

$$\textit{Slave} \qquad \dot{\tilde{\mathbf{x}}} = \mathbf{f}\left(\tilde{\mathbf{x}}, \mathbf{u}, t\right), \qquad (4)$$

where **u** is a periodic function with period *T*, defined by:

$$\mathbf{u} = \begin{cases} \mathbf{x}, & t \in T_{syn} \\ \tilde{\mathbf{x}}, & t \notin T_{syn} \end{cases} \qquad (5)$$

and $T_{syn} \in T$ is the time, during which the synchronization signal from the master system is fed into the slave system.

The principle of this impermanent coupling is shown on Fig. 2. The switch is controlled by the pulse generator.

Many synchronization approaches are proposed so far. In general, they can be divided into two main groups – decomposition ones, by which the master system is decomposed in two parts and one of them is used as a coupling signal; and methods with feedback in the slave system. What is common between all of them, is that each approach permits several variants to design the coupling for a given pair of chaotic systems, but there is no guarantee, that a particular variant of a particular synchronization approach will give stable synchronization for a particular chaotic system. That is to say, no universal synchronization approach exists. Then if one can specify a synchronization method, which retains the advantages of the known methods, but gives much more possible variants to design the coupling, the possibility of finding a stable scheme is greater and the method will be more universal.

By the so called *combined synchronization approach*, proposed by the author, the coupling scheme is designed as a combination between two known synchronization methods-

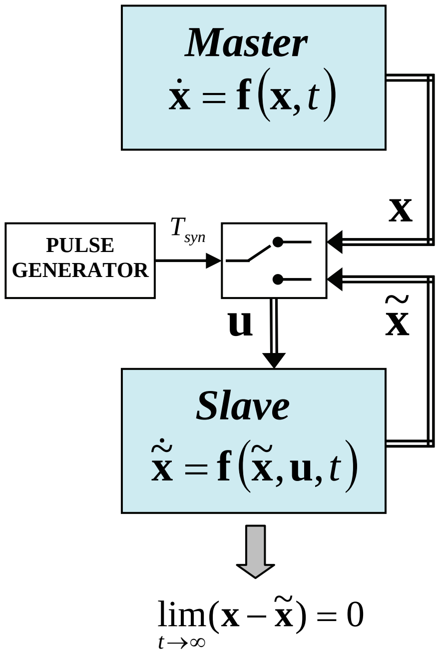

Fig. 2. Synchronization with impermanent coupling

partial replacement [3,8] and *one-way feedback* [1,7]. The master and the slave systems are defined with:

$$\textit{Master} \qquad \dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, x_i\right), \qquad (6)$$

$$\textit{Slave} \qquad \dot{\tilde{\mathbf{x}}} = \mathbf{f}\left(\tilde{\mathbf{x}}, x_i\right) + \alpha\mathbf{E}\left(\mathbf{x} - \tilde{\mathbf{x}}\right), \qquad (7)$$

where x_i is a master system variable, which according to the partial replacement principle substitutes its corresponding slave system variable in only one position of the slave system’s model. Thus several coupling combinations are possible for a given chaotic model. The second coupling is the feedback-type term $\alpha\mathbf{E}\left(\mathbf{x} - \tilde{\mathbf{x}}\right)$ in Eq. (7), where α is the coupling gain and **E** is the coupling matrix, which determines which difference $x_i - \tilde{x}_i$ is introduced in which equation of the slave system’s model.

Apparently, by simultaneously applying the two principles described above, many different coupling variants are possible, much more than those obtained by applying only partial replacement (PR) or only feedback coupling (FC). If for a given system *p* PR variants and *q* FC variants are possible, the combined approach will give *p*x*q* variants. Then one can choose the most appropriate from this vast number of variants, for example the variant with the fastest synchronization, or in terms of the problem discussed here – a variant which uses only one variable if economical coupling is aimed. Additionally, the fastest of all variants with only one particular variable x_i can be chosen.

To make the coupling even more economical, the principle of impermanent coupling can also be applied to the combined synchronization method. The slave system model (7) is then changed by:

$$\textit{Slave} \qquad \dot{\tilde{\mathbf{x}}} = \mathbf{f}\left(\tilde{\mathbf{x}}, u_i\right) + \alpha\mathbf{E}\left(\mathbf{u} - \tilde{\mathbf{x}}\right), \qquad (8)$$

where **u** is defined by Eq. (5).

III. One-Variable Synchronization Scheme for the Hoff Hyperchaotic System

To illustrate the synchronization technique, described in Section II, a well-known hyperchaotic model is used. Hyperchaotic systems possess two or more positive Lyapunov exponents, which define the setting apart of two orbits in state space, started very close to each other. Such systems have more complex behaviour than “regular” chaotic systems with one positive Lyapunov exponent and are preferable in chaotic communication systems.

The Hoff hyperchaotic system [4] is a model of a chemical reactor and is described by the equations:

$$\begin{aligned} \dot{x}_1 &= x_1 - x_1 x_2 - a x_1^2 + 2 n x_4, \\ \dot{x}_2 &= x_1 x_2 - x_2 - b x_2 x_3, \\ \dot{x}_3 &= e + b x_2 x_3 - c x_3, \\ \dot{x}_4 &= a x_1^2 - n x_4, \end{aligned} \qquad (9)$$

where the hyperchaos is most evident for $a = 0.24,\ b = 10,$ $c = 20,\ e = 0.01,\ n = 100$.

Notwithstanding that the Hoff system describes a chemical reaction, its equations can be used as a reliable software chaotic generator with possible application in data-protection systems. To make use of this, one has first to design stable and fast synchronization scheme.

Since the task is to design a one-variable coupling, it will be assumed that only x_1 is accessible. Before the application of the combined synchronization method, the partial replacement and feedback coupling with x_1 are tested alone. It was found, that the best results of five possible x_1-couplings are achieved when this variable substitutes $\tilde{x}_1$ in the fourth equation of the slave system – the transient before the two systems synchronize is about 7 simulation seconds. By applying only the feedback x_1-coupling, the fastest synchronization is for $\alpha = 2$ - about 10 seconds. After consecutively testing all possible x_1-couplings of the combined synchronization method, it was found that the best results are achieved for the following variant of the slave system, designed according to Eq. (7):

$$\begin{aligned} \dot{\tilde{x}}_1 &= \tilde{x}_1 - \tilde{x}_1 \tilde{x}_2 - a \tilde{x}_1^2 + 2 n \tilde{x}_4 + \underline{\alpha (x_1 - \tilde{x}_1)}, \\ \dot{\tilde{x}}_2 &= \tilde{x}_1 \tilde{x}_2 - \tilde{x}_2 - b \tilde{x}_2 \tilde{x}_3, \\ \dot{\tilde{x}}_3 &= e + b \tilde{x}_2 \tilde{x}_3 - c \tilde{x}_3, \\ \dot{\tilde{x}}_4 &= a \underline{x_1^2} - n \tilde{x}_4. \end{aligned} \qquad (10)$$

The systems (9) and (10) are respectively the master and the slave system of the synchronization scheme, designed according to the combined principle of Eqs. (6) and (7). The two systems synchronize for about 4 seconds for $\alpha = 1$ (the fastest result for all possible values of α), as can be seen on Fig. 3, where the error functions $e_i(t) = x_i(t) - \tilde{x}_i(t)$ are presented. The initial conditions are $\mathbf{x}(0) = [3 \quad 3 \quad 0 \quad 0]^T$ and $\tilde{\mathbf{x}}(0) = [4 \quad 1 \quad 1 \quad 1]^T$. The length of the transient does not depend on the initial conditions, so the results are representative for all admissible initial conditions.

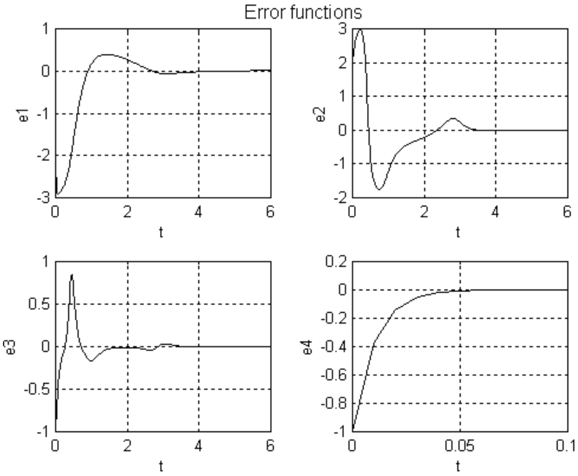

Fig. 3. Error functions for x_1-coupling

The exploiting of the variants with only x_2-, x_3- and x_4-coupling shows, that no stable synchronization exists for every possible only x_2- or only x_3-couplings of the combined synchronization method and the fastest synchronization, achieved for x_4-coupling is with about 7 seconds transient. Thus, the synchronization scheme (9)-(10) is the fastest of all possible one-variable coupling synchronization schemes.

IV. Synchronization with Impermanent Coupling

The x_1-synchronization scheme, defined in Section III is the fastest one-variable coupling scheme for the Hoff system. If the aim is to find not only the fastest coupling with one variable, but to make it more “economical”, if possible, the principle of impermanent coupling, defined in Section II, can be applied to the combined synchronization scheme, defined with Eqs. (9) and (10). To do this, one has to research:

- what is the minimum possible length of T_{syn} to guarantee stable synchronization?
- what is the maximum admissible length of the period T of the driving signal $\mathbf{u}$ from Eq. (8)?

The simulation experiments show, that if $T_{syn} < 7s$, i.e. the x_1-coupling is fed to the slave systems for less than 7 seconds, the systems cannot synchronize.

So, if one choose for example $T_{syn} = 10s$, the next step is to determine a suitable value for the signal period *T*. This is done by applying only one synchronization pulse T_{syn} to the synchronization scheme and let the two systems evolve unconnected to see when they will begin to desynchronize. The error functions for this case are shown on Fig. 4. The coupling is applied only for the first 10 seconds. It is evident, that for the next 70-80 seconds the unconnected systems remain synchronized. Only after that they begin to evolve independent from each other.

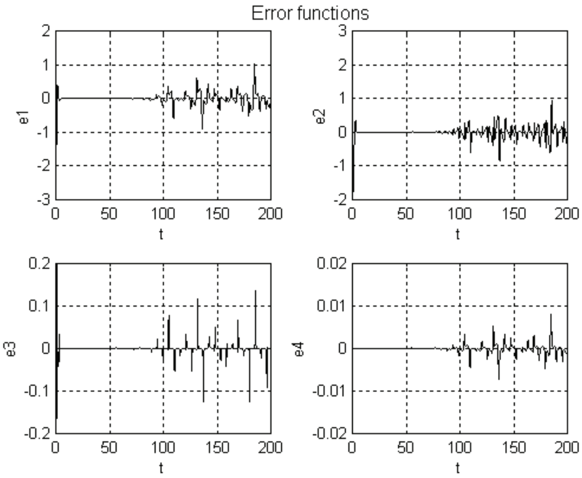

Fig. 4. Error functions with one-pulse coupling

Then, the minimum admissible period of the driving signal **u** which will guarantee uninterrupted synchronization between the master and the slave systems must be $T \approx 80s$ for the case when $T_{syn} = 10s$. The error functions for $T_{syn} = 10s$ and $T = 80s$ are shown on Fig. 5. The moments of activation of the connection between the systems are marked with arrows. The length of the transient before the initial synchronization is achieved is the same as in the permanent-coupling scheme from Section III – about 4 seconds.

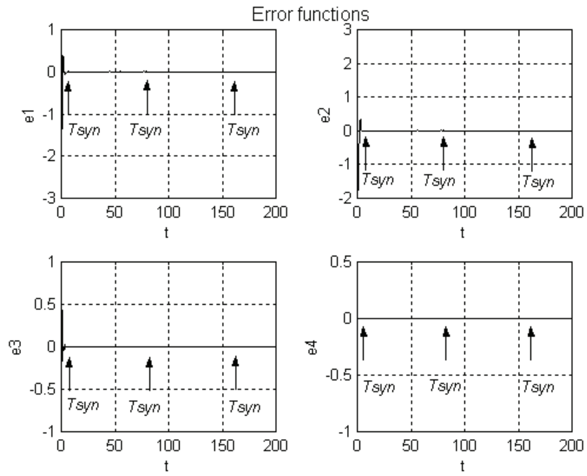

Fig. 5. Error functions with impermanent coupling

It was found however, that the minimum admissible length of the period *T* of the driving signal **u** depends of the value of the synchronization pulse T_{syn}.

For example, if $T_{syn} = 8s$, the two systems run synchronuously only for about 50, not 80 seconds as above, so a suitable value for *T* will be e.g. $T = 40s$. The simulation with $T_{syn} = 8s$ and $T = 40s$ gives exactly the same results as in the previous example.

On the other hand, if $T_{syn} = 16s$, the two systems desynchronize their motion about 150 seconds after the initial synchronization pulse is applied. Then, in order to achieve continuous synchronization, a suitable value for the driving signal's period will be e.g. $T = 140s$.

Thus, the most suitable parameters of the driving signal for any particular synchronization scheme with impermanent coupling can be found only after consistent initial simulation experiments.

V. CONCLUSION

The principle of economic coupling if often aimed in practical chaotic synchronization tasks. The two main aspects of this concept – a one-variable coupling scheme and a non-continuous coupling between the systems, subjected to synchronization, can be applied simultaneously as in the example with the Hoff hyperchaotic system.

First, by applying the combined synchronization approach, the most suitable one-variable synchronization scheme for the particular case is selected. Usually, this is the variant with the fastest synchronization in terms of the possible application for designing a chaotic communication system, where the ability of the chaotic systems in the transmitter and in the receiver to synchronize swiftly if very important. The combined synchronization method offers a vast number of possible combinations for the systems' coupling, so after some initial research the best possible solution for the given case can be found.

Second, by applying the principle of impermanent coupling, one can synchronize two chaotic systems only with short synchronization pulses. There is no need to maintain the systems continuously connected, moreover that this is not always possible in some cases.

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